Hierarchies of theories, Gödel's Programme, and set-theoretic pluralism

Abstract for the GAP12 Logic & Philosophy of Mathematics section

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Since Gödel proved his celebrated incompleteness theorems, the phenomenon of independence has become central in set theory, logic, and the foundations of mathematics. As the case of the Continuum Hypothesis (CH) exemplifies, these issues extend beyond consistency statements and *Gödelian trickery* (e.g., theories like $\mathsf{ZFC} + \neg\mathsf{Con}(\mathsf{ZFC})$) to natural set-theoretic questions with significant mathematical content. The decades following the seminal work of Gödel (1938) and Cohen (1964) have produced a vast amount of independence results, accompanied by a variety of techniques for constructing models and axiomatisations of set theory. This wealth of results carries the philosophical challenge of how to integrate the often incompatible constructions (such as $\mathsf{ZFC} + \mathsf{CH}$ and $\mathsf{ZFC} + \neg\mathsf{CH}$ together with their respective models) into a coherent philosophical picture. One way of meeting this challenge is Gödel's Programme, which was originally introduced in Gödel (1964), and which aims to extend ZFC with axioms strong enough to settle independent statements such as CH. However, the crux of the matter is to find the *right* axioms to add to ZFC.

This brought the problem of axiom selection and axiom justification to the forefront of the foundations of mathematics: which axioms are good enough to be considered axiom candidates, and how to compare them and choose the right extension of ZFC? Several different methodologies have been proposed. The best possible justification, according to Gödel (1947), would be *intrinsic* justification: the axiom has to be necessarily accepted given our intuition of the concept of set. Its nature is intuitively true. Most of the axioms of set theory, Gödel claims, fall into this category. For example, the Axiom of Extensionality or Pairing are both very intuitive and obviously motivated by the iterative conception. On the other hand, the intuitive truth of Martin's Maximum seems less certain, and we have more difficulties in justifying it by appealing to the concept of set. Gödel (1947) argues that, in the cases where intrinsic justification is not enough, we need to fall back to *extrinsic* justification: the new axiom must have very appealing mathematical consequences. The main linchpin of the extrinsic justification is that the new axioms have to be not only with deep mathematical consequences, but also justified by some *external evidence*. Penelope Maddy, in a series of papers and books (see Maddy (1988a), Maddy (1988b), Maddy (1997), Maddy (2011)), has spelled out in more detail what extrinsic justification is with her MAXIMIZE principle: adding a new axiom to ZFC should make the resulting theory as powerful as possible, in the sense that it should *maximize* the range of available isomorphism types. Further refinements of the notion by Löwe (2001), Löwe (2003), and Incurvati and Löwe (2016) have introduced a new notion of interpretability power. To all these notions we must also add the possibility of ordering all the theories by their consistency strength.

What all these methods have in common is that they allow us to arrange the axioms (and consequently the various set theories) in a hierarchical way. With consistency strength this is easily seen: if $\mathsf{ZFC} + \mathsf{A}$ proves the consistency of $\mathsf{ZFC} + \mathsf{B}$, but not the other way around, we say that $\mathsf{ZFC} + \mathsf{A}$ has a higher consistency strength. The same can be done with all the different methodologies methods briefly discussed above (having some care in defining them in mathematical terms, but this is possible). Consequently, we can boil down the goal of Gödel's Programme as follows: find an axiom A such that $\mathsf{ZFC} + \mathsf{A}$ settles the independent questions we are interested in (e.g. CH) *and* it is *maximal* according to one (or more than one) of the justification methods.

In this paper, we argue that such a program is destined to fail. In particular, each justification method gives rise to a sufficiently different hierarchy of theories. For example, consider the case of AD and AC. Mycielski, Steinhaus, and Swierczkowski (1971) proved that they are mutually inconsistent, so our choice is between $\mathsf{ZF} + \mathsf{AD}$ and ZFC. If we try to compare AD and AC by consistency strength, we notice that AD gives us the existence of a measurable cardinal (see Lévy and Solovay (1967)), while AC gives us no large cardinals. However, AC gives us the classic axiomatization ZFC, and it is so fundamental for mathematical practice that hardly anybody prefers to work in $\mathsf{ZF} + \mathsf{AD}$.[1] Consequently, AC is preferable from a naturalist perspective, even if AD is stronger in consistency strength.

For another, more subtle example, consider the debate around the axiom $\mathsf{V} = \mathsf{UltL}$ and forcing axioms like Martin's Maximum (MM). From the perspective of consistency strength, MM sits very high in the hierarchy, between supercompact cardinals and Woodin's cardinal (but the precise consistency strength is still not known, see Foreman, Magidor, and Shelah (1988) and Shelah (1987)). On the other hand, it is conjectured that $\mathsf{V} = \mathsf{UltL}$ has an even higher consistency strength, at the level of I2 or I3 (see McCallum (2018), still unpublished). So in terms of consistency strength, we should prefer $\mathsf{V} = \mathsf{UltL}$ over MM. However, from the perspective of MAXIMIZE, the situation is different: as Schatz (2019) shows, $\mathsf{V} = \mathsf{UltL}$ is *restrictive* over MM, thus implying that, from the naturalist perspective, MM is preferable over $\mathsf{V} = \mathsf{UltL}$.

Moreover, in each hierarchy we can find theories that are equivalent, and thus sit at the same level (for example, we have several equiconsistent theories, see Koellner (2009)). The only solution would be to apply more than one method, but in that case the *order* in which the methods are applied matters, and we still get several different

[1] The arguments usually employed in favour of determinacy are for the axiom $\mathsf{AD}^{L[\mathbb{R}]}$, that it is compatible with AC. See for example Koellner (2009).

hierarchies, with different maximal, incompatible theories on top.

We claim that this a particularly difficult problem for the *universist*, who beliefs that there is only one set-theoretic universe, instantiated by a single set theory. The only solution for them would be to argue that one justification method is better than the other, thus pointing at only one hierarchy, but no progress has been done so far (see for example Barton, Ternullo, and Venturi (2020)). On the other hand, the pluralist has an easier solution: embrace all the different methods, and just include all the maximal theories (or even more than that) to the set-theoretic multiverse.

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