Hierarchies of theories, Gödel's Programme, and set-theoretic pluralism

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Since Gödel proved his celebrated incompleteness theorems, the phenomenon of independence has become central in set theory, logic, and the foundations of mathematics. As the case of the Continuum Hypothesis (CH) exemplifies, these issues extend beyond consistency statements and *Gödelian trickery* (e.g., theories like $\mathsf{ZFC} + \neg\mathsf{Con}(\mathsf{ZFC})$) to natural set-theoretic questions with significant mathematical content. The decades following the seminal work of Gödel (1938) and Cohen (1964) have produced a vast amount of independence results, accompanied by a variety of techniques for constructing models and axiomatisations of set theory. This wealth of results carries the philosophical challenge of how to integrate the often incompatible constructions (such as $\mathsf{ZFC} + \mathsf{CH}$ and $\mathsf{ZFC} + \neg\mathsf{CH}$ together with their respective models) into a coherent philosophical picture. One way of meeting this challenge is Gödel's Program, which was originally introduced in Gödel (1964), and which aims to extend ZFC with axioms strong enough to settle independent statements such as CH. However, the question *which* axioms to add to ZFC remains open until today..

This brought the problem of axiom selection and axiom justification to the forefront of the foundations of mathematics: which axioms are good enough to be considered axiom candidates, and how to compare them and choose the right extension of ZFC? Several different methodologies have been proposed. A first possible justification, according to Gödel (1947), would be *intrinsic* justification: the axiom follows from our conception of set (see e.g. Boolos (1971)). Most of the axioms of set theory, Gödel claims, fall into this category. For example, the Axiom of Extensionality or Pairing are both very intuitive. On the other hand, the intuitive truth of Martin's Maximum seems less certain, and we have more difficulties justifying it by appealing to the concept of set. Besides the intrinsic justification, axioms can also be justified *extrinsically*: the new axiom must have very appealing mathematical consequences. The main linchpin of extrinsic justification is that the new axioms must not only have deep mathematical consequences, but also be justified by some *external evidence.* In a series of papers and books (see Maddy (1988a), Maddy (1988b), Maddy (1997), Maddy (2011)), Penelope Maddy has spelled out in more detail what extrinsic justification is in terms of her Maximize principle: adding a new axiom to ZFC should make the resulting theory as powerful as possible, in the sense that it should *maximize* the range of available isomorphism types. Further refinements by Löwe (2001), Löwe (2003), and Incurvati and Löwe (2016) have introduced a new notion of interpretability power. Finally, an axiom can also be justified in terms of consistency strength: if $\mathsf{ZFC} + \mathsf{A}$ proves the consistency of $\mathsf{ZFC} + \mathsf{B}$, but not *vice versa*, $\mathsf{ZFC} + \mathsf{A}$ has a higher consistency strength and we should prefer A over B.

What all these methods have in common is that they allow us to arrange the axioms (and consequently the various set theories) in a hierarchical way (having some care in defining them in mathematical terms, but this is possible). Consequently, we can boil down the goal of Gödel's Program as follows: find axioms such that, when added to ZFC, the resulting theory settles the independent questions we are interested in (e.g. CH) *and* is *maximal* according to one (or more than one) of the justification methods.

In this talk, we argue that such a program is destined to fail. In particular, each justification method gives rise to sufficiently different hierarchies. For example, the axiom $\mathsf{V} = \mathsf{Ultimate} - \mathsf{L}$ sits higher in the consistency strength hierarchy (according to McCallum (2018), at the level of $\mathsf{I2}$ or $\mathsf{I3}$) than MM. However, the latter sits higher in the hierarchy brought up by Maximize (see Schatz (2019)). Moreover, in each hierarchy we can find theories that are equivalent and sit at the same level, yet they are very different in terms of verifiable consequences and the set-universe they describe (see Koellner (2009)). The only solution would be to apply more than one method, but in that case the *order* in which the methods are applied matters, and we still get several different hierarchies, with different maximal, incompatible theories on top. We claim that this a particularly difficult problem for *universists*, who belief that there is only one set-theoretic universe instantiated by a single set theory. The only solution for them would be to argue that one justification method is better than the other, thus pointing at only one hierarchy, but no progress has been done in this direction (see for example Barton, Ternullo, and Venturi (2020)). On the other hand, the pluralist has an easier solution: embrace all the different methods, and just include all the maximal theories (or even more than that) to the set-theoretic multiverse.

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