A pluralist perspective on Gödel's Programme

Abstract for the workshop "The Limits of Incompleteness"

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The *independence problem* for set theory is a direct consequence of Gödel's Incompleteness Theorems: for any set theory T, we can find a *natural* statement ϕ (i.e. a statement with actual mathematical content, not a self-referential or consistency statement), such that neither $T \vdash \phi$ nor $T \vdash \neg\phi$. The most famous example of such statements is the Continuum Hypothesis (CH), that was proved to be independent from Zermelo-Fraenkel with Choice (ZFC) by Gödel (1938) (that proved that it is consistent with ZFC) and Cohen (1964) (that instead proved that its negation is consistent with ZFC).

The solution most considered in the literature was first introduced by Gödel (1964): find an axiom A that's *sufficiently justified* such that $ZFC + A \vdash \phi$ or $ZFC + A \vdash \neg\phi$, where ϕ is independent from ZFC. The first candidate that were considered as a solution of what is now known as *Gödel's Programme* were Large Cardinals Axioms (LCAs), but it was discovered by Lévy and Solovay (1967) that even the very strong ones don't settle the question of CH. A plethora of other axiom candidates that actually settle CH have been proposed (e.g. forcing axioms like Martin's Maximum, axioms of constructibility like $V = L$, etc.), but they all ended up to be mutually incompatible and no consensus has been reached so far.

The main reason why no consensus has been reached lies in the vagueness of how to "sufficiently justify" an axiom candidate. Several ways to justify axioms have been proposed: intrinsic and extrinsic justification, Maddy's naturalist maxims, looking at interpretability and consistency strength, and so on. However, there is still debate on which kind of justification is the most adapt for our purposes. In particular, all these different justification methods have a common problem: given two axiom candidates A and B, it is possible to find equally legitimate arguments that both A and B are justified and good addition to ZFC.

In this talk I argue that one of the reasons why the debate around what's the proper justification for a set-theoretic axiom candidate is still raging is that such a question is approached from a purely *universist* perspective. That is, the goal is to find *the* right axiomatization that describes *the* set-theoretic universe. Such a perspective is flawed, since it forces a choice that set theorists are unwilling to make: once the right axiom A is found, all the other axioms incompatible with A must be discarded. However, interesting mathematics and philosophy has been done with the other axiom, and it doesn't seem a good choice to discard them as "non-standard".

Another possibility is instead to accept a *pluralist* perspective on the matter. According to pluralism (see Hamkins (2012)), there isn't one true axiomatization that describes the one true set-theoretic universe. Instead, all possible (consistent) axiomatizations and their models can co-exists as legitimate universes of a set-theoretic multiverse. If we follow such a perspective, we can find a different solution to the independence problem and Gödel's Programme: choose the *weakest* and *minimal* set theory that can serve as the ground theory (and its model as the ground universe) of the widest set-theoretic multiverse possible.

References

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