The V-logic multiverse and the Benacerraf's problem

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Abstract

Clarke-Doane (2020) argues that the pluralist stance in the philosophy of mathematics (any consistent mathematical theory produces a legitimate mathematical universe) can provide an answer to Benacerraf problem iff we interpret it in terms of *safety*: our set-theoretic beliefs are reliable iff, for any one of them P, we couldn't have easily had a false belief as to whether P. However, he also argues that it's not clear how the pluralist can show that her set-theoretic beliefs are safe. In this paper, I argue that the V-logic multiverse, a mathematical characterization of the pluralist position, can provides such an answer.

Full proposal

Clarke-Doane (2020) argues that the pluralist stance in the philosophy of mathematics (any consistent mathematical theory produces a legitimate mathematical universe) can provide an answer to Benacerraf problem[1] iff we interpret it in terms of *safety*: our set-theoretic beliefs are reliable iff, for any one of them P, we couldn't have easily had a false belief as to whether P. In other words, if and only if we can be safe that by entertaining that belief we are not easily making a mistake. For example, the belief that "$V = L \wedge \exists 0^{\#}$" cannot be held safely, since we have a proof that it is inconsistent, and we cannot have both the conjunctions. Clarke-Doane (2020) definition of safety is very general, and doesn't provide any insight on what this safety should look like, or how to assess whether a pluralist position actually satisfies it. To do so, I propose the following, more precise, safety principle:

Principle 1 (Safety*)**.** A set theoretic belief φ is *safe* if and only if it is possible to find a theory T such that $T + \varphi$ is consistent, and there exists an extension of V that witnesses such theory.

If we were to entertain a belief that φ, but φ cannot be added consistently to *any* axiomatisation of set theory, then it would be probable that the belief is false, thus not satisfying neither the Safety principle. At the same time, even if the φ could be added consistently to an axiomatisation of set theory, if we still cannot find an extension of V that witnesses this addition we would have doubts on the safety of our belief.

[1] See Benacerraf (1973).

The V-logic multiverse[2] is an extension of Friedman's Hyperuniverse.[3] It is based upon the infinitary V-logic, but it is axiomatized and it doesn't assume the countability of V. For the purposes of the current discussion, the following axiom is of central importance:

Axiom 1 (Multiverse Axiom Schema)**.** For any first-order ψ with parameters from V, if the sentence φ of V-logic expressing "There is an outer model of V satisfying $T + \psi$" is consistent in V-logic, then there is a universe W which is an outer model of $V \models T$ satisfying φ.

This means that each φ consistent in V-logic has a model in the multiverse. This axiom, together with the use of V-logic, allows the pluralist to provide a good answer to Clarke-Doane (2020) problem. Suppose that a set theorists has the belief φ, and wants to know whether this belief is safe. To do so, in the V-logic multiverse, she searches for a theory T consistent in V-logic. Then it is possible to use the infinitary proof system of V-logic to check if φ is satisfied in some extension of V. Starting with premises T, we derive φ in V-logic. This gives us a proof that the theory $T + \varphi$ is consistent in V-logic.[4] By applying the Multiverse Axiom Schema, we then know that *there exists* an extension W of V in the V-logic multiverse such that $T + \varphi$ is satisfied in that extension, $W \models T + \varphi$. If this is the case, then we can conclude that the belief that φ is indeed safe: we managed to find a consistent theory that includes φ *and* we found an extension of V that satisfies this theory. As an example, we can apply this method to the CH. In this case, it is easy to find out that the theory $ZFC + CH$ is indeed consistent in V-logic. Thus, there exists an extension W in the V-logic multiverse, produced by forcing, such that $W \models ZFC + CH$. So, our belief that CH is safe. We couldn't have easily a false belief on this: any φ such that $T + \varphi$ is consistent in V-logic is indeed safe, with an extension of V that satisfies it. For example, it wouldn't be possible to have the belief that $CH \wedge \neg CH$, since such φ wont be consistent with any theory T in V-logic (for a more interesting example, consider the sentence "$V = L \wedge \exists 0^{\#}$".

References

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[2] See de Ceglie and Ternullo (n.d.).

[3] See Antos, Friedman, et al. (2018).

[4] For the technical details see Antos, Barton, and Friedman (2021).