The Multiverse Operator

A generalisation of the set theoretic multiverse

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Ten years ago, Hamkins, 2012 changed the landscape of the foundations of mathematics, by introducing a novel conception that tried to clarify some ambiguous notions in current set theoretic practice. In particular, he provided a revolutionary interpretation for the practice of forcing: a multiverse of different set theoretic universes. Such an idea immediately sparked an intense debate in the philosophy of set theory and the foundations of mathematics. In the following years, several crucial contributions were made by Antos, 2018, Bagaria and Claudio Ternullo, 2020, S. Friedman, 2012, Gitman and Hamkins, 2011, Koellner, 2009, Maddy, 2017, Meadows, 2021, Martin, 2001, Steel, 2014, C. Ternullo, 2019, Väänänen, 2014, Venturi, 2020, and Woodin, 2011, just to name a few. These contributions can be roughly divided in two broad categories:

- the general debate between *universism* (the position that there is a single, determined universe of sets) and *pluralism* (the position that there are several universes of sets, all of them equally interesting, i.e. the multiverse);[1]
- the introduction of novel mathematical characterisations of the set theoretic multiverse.[2]

Indeed, while the general idea behind pluralism in the philosophy of mathematics is more or less the same every time, the actual mathematical details can vary enormously from one characterisation to the other. We have multiverses based upon different kinds of forcing[3] , multiverses with different background logics[4], multiverses that try to accommodate the highest number of different universes[5], etc. Even though all these different set theoretic multiverses share the same, general, philosophical idea, they differ wildly from the mathematical perspective. There are some proposal of assessing all these differences[6], but this research field is still in its infancy. In this paper, I propose a novel, more general, framework for the set theoretic multiverse: the *Universal Multiverse*.

Drawing from the ideas of Universal Logic[7], this new framework is *not* a new set theoretic multiverse. Instead, it is a *general theory of the multiverse*, that investigates the various set theoretic multiverses from a common and abstract point of view. This means highlighting the common features of all multiverses, studying them as mathematical structures without any other metaphysical and ontological connotation.

The first step to carry out this program is to define a method of comparing and categorising all the different multiverses. A multiverse can be characterised in several ways: the most common one is to describe it as a set of universes. For example, Steel's set generic multiverse is the set of all set-generic extensions of a core universe, Friedman's Hyperuniverse is the set of all countable transitive models of ZFC, etc..

Generalising on this idea, a general theory of the set theoretic multiverse is the theory of the *Multiverse Operator*, Mlt_i. This theory is analogous to Tarski's theory of the consequence operator as introduced in Tarski, 1928. A multiverse operator is a function defined on the powerset of a set S of sentences. This set S is the set of all possible set theoretic axioms, so each set theory T is a subset of $\mathcal{P}(S)$. A multiverse operator maps each theory T with the set of all universes, M, that are part of the multiverse generated from T:

$$Mlt_i : T \mapsto M.$$

For example, the operator $Mlt_{generic}$ applied to the set of sentences MV (the multiverse theory from Steel, 2014), written $Mlt_{generic}(MV)$, will map to Steel's set generic multiverse. Another example is the operator

[1] Examples of such papers are Koellner, 2013, Martin, 2001 and Maddy, 2017.
[2] For example, Gitman and Hamkins, 2011, Steel, 2014 and C. Ternullo and S.-D. Friedman, 2016.
[3] For example, Steel, 2014 is based upon set-generic forcing, while Venturi, 2020 on Robinson infinite forcing.
[4] Väänänen, 2014 and S. Friedman, 2012 are the prime examples.
[5] Hamkins, 2012 is the maximal multiverse conception, encompassing all possible universes.
[6] See for example Meadows, 2022.
[7] See Beziau, 2007.

Mlt_{vlogic}, that maps ZFC to the set of all outer models of V (not only the set-generic ones).[8] In this way we can define in very general terms what a set theoretic multiverse is: it is an ordered couple (T, Mlt_i), where T is a set of axioms and Mlt_i is a multiverse operator.

The general theory of the multiverse that I propose is the study of the multiverse operator in *general*, without considering the special feature of each of them (e.g. the difference between $Mlt_{generic}$ and Mlt_{vlogic}). From this very abstract and general perspective, I claim that the general structure $\langle S, Mlt \rangle$ forms a *Tarski structure*. That is, it obeys the following axioms:

1. $T \subseteq Mlt(T)$;
2. $T \subseteq U \implies Mlt(T) \subseteq Mlt(U)$;
3. $Mlt(Mlt(T)) \subseteq Mlt(T)$.

In this paper, I will expand on the meaning of these axioms for the multiverse operator, and sketch the road forward in this research field.

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[8] One of the rendition of such a multiverse is Friedman's Hyperuniverse Program from S. Friedman, 2012.