The V-logic Multiverse and MAXIMIZE

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In this paper, I argue that classical set theory, $ZFC(+LCs)$, is *restrictive* over the V-logic multiverse (a novel set theoretic multiverse conception developed by the present author and Claudio Ternullo). This multiverse conception is based upon Friedman's hyperuniverse and Steel's set-generic multiverse: like the hyperuniverse, it uses the infinitary V-logic as the background logic and admits all kinds of outer models of V (produced by set-generic, class-generic, hyperclass forcing, etc.). Like Steel's set-generic multiverse, it is recursively axiomatisable and is rooted on a ground universe that satisfies ZFC. For this proof, I compare $ZFC + LCs$ and the V-logic multiverse, characterised as $ZFC + LCs+$ the multiverse axioms, following Maddy's methodological principle MAXIMIZE, as described in Maddy (1997). According to this principle, when choosing between two foundational theories we ought to prefer the one that can prove more isomorphisms types. I claim that the V-logic multiverse, as opposed to $ZFC + LCs$, does exactly that. This is because in the V-logic multiverse theory we can prove the existence of proper, uncountable, extensions of V, that we cannot have in $ZFC + LCs$. In turn, this extra object means we can realise more isomorphisms types, since in the V-logic multiverse we can prove the existence of iterable class sharps and, more importantly, maps between them. This opens up a new realm of isomorphisms types that are not available in $ZFC + LCs$. Thus, this latter theory is restrictive over the V-logic multiverse theory: the V-logic multiverse, characterised as $ZFC + LCs+$ Multiverse Axiom Schema, and with V-logic as the background logic, proves more isomorphism types than classical set theory ($ZFC + LCs$), and thus we can say that classical set theory is *restrictive* over it.

First of all I need to precise the terms of this comparison. On the one hand, I am taking classical set theory in its usual axiomatization ZFC plus the addition of large cardinals axioms, as instantiated by the the cumulative hierarchy V. This, together with the usual interpretation of forcing as applicable only to countable transitive models is the foundational framework that universists defend, and claim that it is enough for set theoretic practice. On the other hand, the V-logic Multiverse is characterised as $ZFC + LCs+$ the Multiverse Axiom Schema (there are also other axioms, but they are not really needed right now). Note that, as usually argued by the universist, the addition of the Multiverse Axiom Schema does not add any "real" power to $ZFC + LCs$, since everything we need is already in the latter theory, at least according to universists.

We can now proceed to the first step of my argument, i.e. proving that the V-logic multiverse can prove the existence of an extra object that it is unavailable in $ZFC + LCs$. This object is a proper, uncountable, outer model of V. Such an object cannot exists in the universist's framework of $ZFC + LCs$: indeed, the application of forcing in that usual setting is done only to countable transitive models. This is because to do it we need the existence of generic filters, and for the universist there are no V-generic filters.

However, in the V-logic multiverse framework we can prove the following theorem:

Theorem 1. *Let φ be a V-logic sentence (for instance, a sentence which says "$Con(T)$" for some V-logic theory T). The following are equivalent:*

1. *φ is consistent in V-logic.*

2. φ is consistent in $\mathfrak{V}$-logic.

3. $\mathfrak{V}$ has an outer model, $\mathfrak{W}$, such that $\mathfrak{W} \models \varphi$.

4. (since $\mathfrak{W}$ is elementarily equivalent to $\mathfrak{V}^*$) $\mathfrak{V}^* \models \varphi$.[1]

What this theorem says is that, in the V-logic multiverse, even if we start with a countable model of ZFC inside V, we can then end up with a proper, uncountable outer model of an *uncountable* V[2]

Consequently we have, in the V-logic Multiverse, an object that cannot be found in the universist's framework. We now need to prove that this new object realises a new isomorphism type. And this is exactly my claim.

To see this, consider the technique of #-generation.[3] As stated by Antos, Barton and Friedman, this method is very useful in encapsulating several large cardinals consequences of reflection properties. It is based upon the existence of *class-iterable sharps*: these are transitive structures that are amenable, with a normal measure and iterable in the sense that all successive ultrapower iterations along class well-orders are well founded.[4] If such an object exists, then we could have *class iterated sharp generated* models, i.e. models that arise through collecting together each level indexed by the largest cardinal of the model that result from the iteration of a class-iterable sharp.[5] Finally, we could claim that V is such class iterably sharp generated, and enjoy all advantages of this fact. Sadly, we cannot, since we cannot find, in V, a class-iterable sharp. If it were the case, then we would be able to prove the existence of a cardinal that is both regular *and* singular, but this is impossible. So in the classical set theoretic framework all of the above is unattainable.

This situation is fundamentally different in the V-logic multiverse. Indeed, since in the V-logic multiverse we can have proper, uncountable, extensions of V, we can also have, in these extensions, a class-iterable sharp! And thus, in the V-logic multiverse, we can claim that V is, in fact, class iterably sharp generated! This result opens a new realm of isomorphisms types between all the various iterated ultrapowers, and models of different heights that are provided by #-generation.

Thus, in conclusion, we can claim that $ZFC + LCs$ is restrictive over the V-logic multiverse, since in the latter we can find a new object that realises a new isomorphism type.

References

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[1] This theorem has been proved by the present author and Claudio Ternullo in the paper *Outer models, V-logic and the Multiverse*, currently in preparation, and it is based on similar results from Antos, N. Barton, and S.-D. Friedman (nd) and Neil Barton (2019).

[2] The V-logic multiverse is not the only multiverse conceptions that claims the existence of proper outer models of V, the other being the Hyperuniverse. However, the latter assume that V is countable, thus semplifying the setting by a lot.

[3] See Antos, N. Barton, and S.-D. Friedman (nd) for a discussion of it.

[4] Here I am following the definition from Antos, N. Barton, and S.-D. Friedman (nd). The original definition in S. Friedman (2016) is slightly different, however nothing important rests on this difference.

[5] Again, the precise definition can be found in Antos, N. Barton, and S.-D. Friedman (nd).