

Beyond Naturalism: Mathematical Neutrality in the foundations of mathematics

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In this paper I argue that all methods introduced to justify new axioms to achieve Gödel’s Programme (namely, intrinsic and extrinsic justification, Maddy’s naturalism) have problems when applied to cases in which the axioms compared are mutually incompatible but mathematically equally interesting (for example, in the case of $V = \text{Ultimate} - L$). This is because they all assume a universalist, and maximal, perspective on the solution to Gödel’s Programme: only one set-theoretic axiomatisation can serve as a foundation of mathematics, and we need to find the “best” one that allows us to prove as much as possible (modulo incompleteness). However this doesn’t work as intended, since when evaluating these incompatible axioms it’s possible to find arguments supporting both sides. In this paper, I propose a new principle: NEUTRALITY, derived and inspired both from Maddy’s Naturalism and Shelah’s reflections on set-theoretic practice. I will argue that even this principle, if applied from the universalist perspective, cannot aid us in finding a solution to Gödel’s Programme. On the other hand, NEUTRALITY can support a minimal, pluralist solution to Gödel’s Programme: a pluralist framework in which several set theories can coexist can serve as a functioning foundation of mathematics. For these reasons, I argue that its addition to the other methods is a worthy one, and can help in making progress in the foundational debate of axiom selection and justification.

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1 Introduction

Since Gödel proved his celebrated incompleteness theorems, independence has become a central phenomenon in set theory, logic, and the foundations of mathematics. As the case of the Continuum Hypothesis (CH) exemplifies, these issues extend beyond consistency statements and *Gödelian trickery* (e.g., theories like $ZFC + \neg\text{Con}(ZFC)$) to natural set-theoretic questions with significant mathematical content. The decades following the seminal work of Gödel (1938) and Cohen (1964) have produced a vast amount of independence results, accompanied by a variety of techniques for constructing models and axiomatisations of set theory. This wealth of results carries the philosophical challenge of how to integrate the often incompatible constructions (such as $ZFC + \text{CH}$ and $ZFC + \neg\text{CH}$ together with their respective models) into a coherent philosophical picture. One way of meeting this challenge is Gödel’s Programme, which was originally introduced in Gödel (1964), and which aims to extend ZFC with axioms strong enough to settle independent statements such as CH, while leaving the resulting theory a good foundation of mathematics. However, the question *which* axioms to add to ZFC remains open.

This raised the problem of axiom selection and axiom justification to the forefront of the foundations of mathematics: which axioms are good enough to be considered axiom candidates, and how to compare them and choose the right extension of ZFC ? Several different methodologies have been proposed, from intrinsic and extrinsic justification, to Maddy’s naturalism. However, Gödel’s Programme has seen little concrete progress, and the very question on whether it is possible to accomplish his goal of expanding ZFC is still unanswered. Mathematical *pluralism* offers a recent approach to this challenge. In the set-theoretic context, pluralism morphs into *multiversism*, which posits that there are many different set-theoretic universes, all of which are all ontologically on a par.

In this paper, I argue that all the methods investigated so far (intrinsic and extrinsic justification, Maddy’s naturalism) fail to give us answers when comparing mutually incompatible, but mathematically equally interesting, axioms. To avoid this impasse, I propose a novel methodological principle: the NEUTRALITY principle. According to this new principle, the better foundation of mathematics is the one that restricts less the kinds of mathematical objects and proof methods available to the working set theorist. As I claim, this principle is compatible with Maddy’s naturalism and should be added to our methods of axiom justification, to be used when the other methods don’t give us a definite answer. However, I contend that also NEUTRALITY cannot help support a definite answer to Gödel’s Programme for the same reasons why the other methodologies cannot. Instead, I argue that it supports a *minimal* approach to the problem of axiom

selection that favours a pluralist solution to Gödel’s Programme for set theory.

The paper is organised as follows. First, I give a brief account of Gödel’s Programme and the main methods developed to accomplish it (section 2). In particular, I focus on intrinsic and extrinsic justification (section 2.1) and Maddy’s naturalism (section 2.2), and the problem we face when applying them to some concrete cases. Second, I introduce the NEUTRALITY principle (section 3). After a general presentation of the principle (section 3.1), I present some problems when approached in the same way as the other methods (section 3.2). Finally, I argue that it supports a pluralist framework for the foundations of mathematics 4. Some final remarks conclude the paper (5).

2 Possible solutions to the problem of axiom justification

2.1 Gödel’s Programme and the problem of axiom justification

Gödel’s Programme aims to add new, justified axioms to ZFC, to make it more powerful and settle the independent questions, especially the CH (Gödel, 1947). This is the core of Gödel’s Program: since we know that ZFC is not powerful enough to settle the independent CH, an extensions of ZFC, meaning ZFC with a new added axiom A, might be powerful enough to do it. This leaves us with the *axiom selection* problem: how do we choose a suitable axiom A? According to Gödel, 1947, the axiom candidate A needs to be strongly justified, and proofs with A (of facts already known) should become easier and better. If we manage to find such an A, we may then argue that ZFC + A is a good foundation of mathematics, that extends the good foundationality features (at this point, still vaguely defined) of ZFC with the new powerful methods provided by A. The main hurdle here is in how we justify the addition of A, so the axiom selection problem becomes in reality an axiom *justification* problem.

Gödel (1947) argues that *intrinsic* justification is ideal: the axiom has to be necessarily accepted given our intuition of the concept of set. Its nature is intuitively true. Most of the axioms of set theory, Gödel claims, fall into this category. For example, the Axiom of Extensionality or Pairing are both very intuitive. On the other hand, Martin’s Maximum¹ appears less intuitively truth, and we have more difficult in justifying it by appealing to the concept of set.

However, intrinsic justification has its limits, and doesn’t work as intended. The main problem resides with the notion of “self-evident” truth that an axiom should display to be considered intrinsically justified, and with the appeal to “intuition” that is also appended to this kind of justification. The hope when dealing with competing axiom candidates is that one appear clearly and intuitively true, perfectly aligning with our concept of set. This, however, runs into some problems, as argued by Maddy (1988a) and Fontanella (2019) (among others). First of all, our concept of set doesn’t pin down all the necessary features of it. We usually appeal to analogies and metaphors to explain

¹Martin’s Maximum is a generalisation of the Proper Forcing Axiom, and it represent the broadest class of forcings for which an axiom is still consistent. Very roughly, it states that if D is a collection of \aleph_1 dense subsets of a notion of forcing that preserves stationary subsets of ω_1 , then there is a D -generic filter. See Foreman, Magidor, and Shelah (1988) for more details.

what a set is (for example, in basic textbooks on set theory a set is generally described as a "collection" or a "box"), but this is not helpful in more limit cases. Case in point: when approaching the independent problem, our concept of set can be used to argue back and forth between two opposite (and mutually incompatible) positions, without making any progress. For example, it might be argued that it is intuitively true according to our concept of set that there are no subsets of \mathbb{R} with cardinality less than \mathbb{R} but more than \mathbb{N} (thus supporting the truth of CH). Such a position would support the intrinsic justification of any axiom that entails CH. However, the opposite position seems also intuitive enough: why shouldn't we be able to find a subset of \mathbb{R} that's bigger than \mathbb{N} but less than \mathbb{R} ? Indeed, \mathbb{R} seems big enough to contain such a subset! Consequently, we end up arguing that all the axioms that entails the negation of CH are intrinsically justified, essentially with the same argument as before, and thus we make no progress.

Another problem with intrinsic justification is that it might possible to appeal to *different* concepts of set (implicitly or explicitly). As Incurvati (2020) argues, our conception of set is not that monolithic and clear cut. While the standard conception is the iterative conception, that justifies most, if not all, of our axioms with the picture of the cumulative hierarchy, there are other, competing conceptions that fare equally well in justifying the axioms (the same ones or even different ones). For example, while the iterative conception justifies the Axiom of Foundation (that states that no set can be member of itself), a *graph* conception of sets, in which sets are seen as rooted trees visualising the membership relation between their members, seems to justify a negation of the Axiom of Foundation, for example Aczel's Anti-Foundation Axiom (see Incurvati (2014)). The resulting set theory is a workable foundation of mathematics. The two axioms are "intrinsically unjustified" from the perspective of the wrong conception of set, but become perfectly justified under the right one. Such a difference seems to entail that the axioms we consider intrinsically justified will be different according to which conception of sets we consider the right one. Consequently, before using intrinsic reasons to justify axioms we need to define under which conception of set we are operating. And while this might seem obvious, there are actually difference that are not immediately solvable. This leads to the problem that the main linchpin of intrinsic justification is the philosophical arguments in favour and against particular conception of sets, that in turn decide the way we decide how to apply intrinsic justification. As Maddy (1997) argued, this is not as precise as we would want in the mathematical context.

A further problem that Maddy (1988a) points out is that it is not always clear that when intrinsically justifying an axiom we are appealing *only* to intrinsic reasons, i.e. its intuitive truth given a particular conception of sets. Instead, Maddy's argues that other methodologies, like an appeal to its benign consequences (i.e. extrinsic justification, see below), play a (even bigger) role even in officially intrinsic arguments. Even in cases in which the justification seems superficially intrinsic (e.g. the justification of the Axiom of Extension), it turns out that the core of the arguments actually relies on extrinsic reasons. For all these reasons, we need a more mathematical method to supplement intrinsic justification.

Gödel (1947) argues that, in the cases where intrinsic justification is not enough, we need to fall back to *extrinsic* justification: the new axiom must have very appealing

mathematical consequences. The main linchpin of the extrinsic justification is that the new axioms have to be not only with deep mathematical consequences, but also justified by some *external evidence*, as it is clear from the following quote:

[...] success here means fruitfulness in consequences, particularly in “verifiable” consequences, i.e. *consequences verifiable without the new axiom*, whose proof with the help of the new axiom, however, are considerably simpler and easier to discover, and which make it possible to contract into one proof many different proofs [...] There might exist axioms so abundant in their verifiable consequences, shedding so much light on a whole field, yielding such powerful methods for solving problems [...] that, no matter whether or not they are intrinsically necessary, they would have to be accepted at least in the same sense as any well established physical theory.[Gödel (1947, p.477), emphasis added]

To illustrate the kind of arguments that Gödel’s Programme entails, the debate around the addition of the Axiom of Choice (AC) to ZF is quite illustrative (even though historically was already settled at the time Gödel proposed his justification methods). First of all, remember that the Axiom of Choice states (roughly) that there exists a choice function that picks one element from each set of a family of sets, even if both the sets and the family of sets are infinite. It was first used by Zermelo (1904) to prove the Well-Ordering Theorem (i.e. that every set, even the real numbers and other uncountable sets, can be well-ordered), and then officially stated in Zermelo (1908). AC cannot be justified by appealing to its nature, and when was first proposed by Zermelo (1908) it sparked a controversial debate. The main problem with AC was its non-constructive nature: it stated the existence of a function *without* giving any method to actually build that function. For Brouwer (see Brouwer (1910)) and the intuitionist this was too big of a flaw in the axiom. Moreover, it also led to some results that were viewed as paradoxical, for example the Banach-Tarski theorem (see Banach and Tarski (1924)). The question about AC however remained: is it a worthy addition to ZF? Appealing to the nature of the concept of set wasn’t helpful: the intuitionists argued that, since the axiom didn’t provide an actual constructed method for the choice function, the axiom wasn’t intuitively true. On the other hand, if we don’t really care about constructivism and intuitionism, and instead turn to the consequences of AC, it is immediately clear that its consequences are central to several mathematical branches. It suffices to mention the Kuratowski-Zorn Lemma (Kuratowski (1922) and Zorn (1935)) and the Well-Ordering Theorem (already proved in Zermelo (1904)). A mathematics without the AC would have meant a severe restriction of available results, and thus the AC was in the end accepted.² Moreover, adding AC to ZF makes a lot of proofs, that wouldn’t need it, easier (for example, it makes the proof of the Schröder-Cantor-Bernstein Theorem nearly trivial). This looks like a prototypical application of extrinsic justification: the addition of *maths* AC needed justification, and while the debate around its nature and intuitive truth was raging, it became so central to mathematics for its consequences that was *de facto* justified.

²For a better and more detailed exposition of the history behind the acceptance of the AC see G. H. Moore (2012).

Even if extrinsic justification is already better than intrinsic justification, it still has some problems in its application. The main problem is that the idea of what counts as a principled preference of some consequences over others is difficult to properly pin down: how do we measure it? There should be a reliable method to compare axiom candidates in their consequences, and check which one has the better ones. But this can only be done holistically: we look at all the various consequence, and judge which one seems to be better for set-theoretic practice. While this is not a bad method *per se*, a better one is still needed. The reason is that mutually incompatible axioms could have equally interesting mathematical consequences. For example, consider the axioms $V = Ultimate - L$ and the Proper Forcing Axiom (PFA).³ The former implies that we can have *all* large cardinals in an inner model, and moreover it settles CH for its truth. However, PFA also settles CH (for its falsity), and it allows us to go wild with forcing constructions of any type. In both cases, the mathematics we can do is very interesting, and has sparked very fruitful research programs and fields. Consequently, it is not possible to look at those two axioms holistically and point to one of them having the “better” consequences. It would be better if we had a proper, mathematical way to compare these axioms (e.g. by consistency strength), but this also gets us nowhere (in the case of consistency strength, we end up with a bunch of theories that are all equiconsistent, that we then need to compare in some other way).⁴

Gödel thought that intrinsic and extrinsic justification were enough to justify any new axiom. His hope was to extrinsically justify the Large Cardinals Axioms so that they could be accepted as a solution to CH. However, his hope was not fulfilled: it was later discovered (see for example Lévy and Solovay (1967)) that LCAs could not settle CH. Moreover, a new type of axioms started getting investigated: forcing axioms.⁵ These axioms allowed certain kinds of forcing to be carried over, and settled CH in the negative way. For example, the Proper Forcing Axiom (PFA) implies the negation of CH, and that the continuum has size \aleph_2 . Such a solution is opposed to that of *constructibility* axioms, like $V = L$, $V = L[\mathbb{R}]$, \dots , that instead settle CH positively (Gödel (1944)). However, they also contradict the existence of most large cardinals (Scott (1961)). Thus there are two collection of axioms, that have very different natures and very different consequences. Arguments that appeal to intrinsic or extrinsic justification cannot help us here, so we need to further specify our methods of justification. This is where Maddy’s naturalism enters the picture.

2.2 Maddy’s naturalism

As we saw in the previous section, intrinsic and extrinsic justification, while useful in giving us a first line of argumentation when trying to add new axioms to ZFC, are limited in the kind of justification we can achieve. As we saw in the previous section, when trying to asses possible axiom candidates to extend ZFC (e.g. LCAs, $V = L$,

³Forcing Axioms gives us conditions on which certain types of forcing are possible or not. In particular, PFA states that proper forcing is possible, see J. T. Moore (2010).

⁴However, this is a problem worthy of its own treatment that cannot be carried out here.

⁵For a survey of forcing axioms, see Burke (1998).

forcing axioms, etc.), neither intrinsic nor extrinsic justification can help us making real progress in the debate. Consider again the case of LCA. There are arguments that consider LCA intrinsically justified on the grounds of ZFC only and our iterative conception of set theory. However, counter-arguments point out that such an intrinsic justification requires some kind of maximality principle (i.e. “the universe is as high as possible”) or some reflection principle (e.g. Uniform Reflection, or the Global Reflection Principle). Thus, the intrinsic justification of LCA requires resources beyond ZFC, and thus they are not intrinsically justified. How to decide which side of the debate is right (if it is at all possible)? The same problems arise in the case of extrinsic justification: both axioms of constructibility (e.g. $V = L$) and Forcing Axioms (e.g. PFA) have nice and appealing mathematical consequences, but they are mutually incompatible. It seems that any side of those arguments can equally appeal to intrinsic and (especially) extrinsic justification. For this reason, we need some new method to either support intrinsic and extrinsic justification, or replace them.

A possible way to approach the question is to let set-theoretic practice speak for itself: set-theoretic methods, as used by working mathematicians, are the only way to approach questions regarding the foundations of mathematics. To investigate which axiom to add, we can only appeal to the actual set-theoretic practice. Such an approach has been first advocated by Quine (1971), and later developed in detail by Maddy in a series of articles and books (Maddy (1988a), Maddy (1988b), Maddy (1996), Maddy (1997)). Maddy’s philosophical perspective arises from the need for a better method for choosing new axioms for set theory. In particular, one of her first case studies was the choice between $V = L$ and the existence of a measurable cardinal.

However, as we saw, also appealing to the consequences of an axiom could be quite debatable: a particular consequence could be needed by some mathematicians but not by others, or two opposite axioms could have equally appealing mathematical consequences.

To improve intrinsic and extrinsic justification as a methodological tool, Maddy widens their scope: when adding a new axiom to ZFC, one is not only extending our base set theory, but also extending the foundation of mathematics.⁶ Consequently, when trying to justify the addition of a new axiom, one cannot look only at its mathematical consequences, but also at the foundationality relevance of the new theory.

First of all, Maddy (1997) notices that the main point of a foundational theory is to provide a shared framework for all mathematical practice (and not only set-theoretic practice; however, we will follow Maddy in focussing on only the latter):

Principle 1 (UNIFY). *The ultimate goal of a foundational framework is to provide a common framework where every structure and every mathematical object can be modelled.*

This essentially says that an adequate set theory should allow one to model most (better: all) of current mathematical practice. Classical set theory as axiomatised by ZFC clearly fits the bill: all of mathematics can be seen as being done “inside” set theory. Or, in

⁶This is assuming that set theory (in general) is the foundation of mathematics. There are also other candidates, for example Homotopy Type Theory/Univalent Foundations (HoTT/UV) or category theory. They will come in play later in the section.

other words, all mathematical objects, constructions, proof methods, and so on, can be reduced to sets and set-theoretic methods.

Maddy (2017) further refines UNIFY, saying that a theory T is foundational (and satisfies UNIFY) if and only if it provides the following:

Meta-mathematical Corral A foundational theory should be able to encompass all possible mathematics and function as a good meta-theory.

Elucidation The foundational framework T should be able to replace a vague notion of some mathematical object with a more precise one.

Shared Standard A foundational theory should provide a definition and benchmark of what counts as “proved”.

Risk Assessment A foundational theory should provide tools and methods to check if a mathematical object is consistent and safe.

Generous Arena A foundational theory should provide a mathematical universe in which all mathematics can take place.

In other words, if in the new theory we can prove something general about all of mathematics, clarify mathematical concepts⁷, provide the basis of what it means to be “proved” in mathematics, and allow us to investigate the consequences of adopting a certain axiom or object, then it is foundational. Finally, Generous Arena means that all the various structures studied in all mathematical branches ought to co-exist in the foundational framework, so that their relations can be studied in a common environment. Moreover, our foundational framework should be a place where is possible for a mathematical field to borrow methods from another field.

Maddy (1997) also argues that one might compare all these different theories according to the power of their proof methods, and their “ability” to encompass more mathematical objects. In other words, this means preferring a theory that allows us to prove more theorems, and that includes more mathematical objects. This attitude is condensed in the principle MAXIMIZE:

Principle 2 (MAXIMIZE). *A foundation of mathematics should be as powerful as possible, that is it should maximise the range of available isomorphism types.*

The intuitive idea behind this second principle is that a foundational framework should allow mathematicians to compare different objects as described in different theories and their relationships. The more isomorphism types a theory can prove, i.e. the more relations of “structural identity” between structures and objects can be established, the more a theory can fulfil this foundational goal. When comparing two theories, the one that

⁷This is how set theory was developed by Cantor and Dedekind, trying to clarify the notion of continuity which, before Dedekind’s work, was somewhat ambiguous. It was precise enough to generate the calculus, but not precise enough to become a tool in the proof of fundamental theorems. Only after it was defined in set-theoretic terms was the notion sufficiently clarified so as to be employable in any area of mathematics.

proves more isomorphism types is preferable. The theory that proves more isomorphism types *maximises* over the other (or, equivalently, the theory that proves less isomorphism types is *restrictive* compared to the other). Maddy (1997) then applies this principle to argue against the addition of $V = L$ to ZFC, and in favour of adding LCAs instead.

Armed with these new methodological principles and justification methods, we can try to apply them and see whether the situation has improved over the case with only intrinsic and extrinsic justification. Category theory, a possible alternative to set theory as a foundations of mathematics, claims to fulfil UNIFY just as ZFC fulfils it, while taking as primitive notions not the notions of set and membership, but the ones of category and isomorphism.⁸ And indeed, it is true that all mathematics can be done inside category theory, even set theory (Blass (1984)). It is also true that category theory can be simulated inside set theory (see for example Barr et al. (1969)). So it seems that the two alternative foundations are equivalent. While historically ZFC has always been the first choice, it can be argued that this is only for historical precedence, having been developed before category theory. But this leaves open that category theory is actually the better foundation. So how do we compare them?

Maddy (2017) argues that ZFC (plus Large Cardinals), when construed as the theory of the standard cumulative hierarchy V , fulfils all the foundational roles needed (i.e. Meta-mathematical Corral, Elucidation, Shared Standard, Risk Assessment, and Generous Arena). And indeed this is true, since they are tailored around current set-theoretic practice (that is carried out mostly in ZFC). More importantly, Maddy (p.29 2017, p.34) argues that category theory *doesn't* fulfil all these foundational roles compared to ZFC(+LCAs). For this reason, Maddy argues, set theory is a better foundations than category theory. Maddy (2017) considers Univalent Foundations only in passing, when discussing Meta-mathematical Corral and Risk Assessment. After that, she doesn't assess the Univalent Foundations program using her foundational roles like she does with category. However, it might be argued that similar arguments to the ones put forward against category theory could also be employed in the case of Univalent Foundations.⁹

Having established that ZFC + LCAs is the best possible foundations of mathematics, a new question arises. ZFC + LCAs is not the only possible set theory we can use as a foundations of mathematics. We might prefer ZF + AD, or ZFC + $V = L$, or ZFA (ill-founded set theory, see Aczel (1988)). All of these different set theories are still based upon the Zermelo - Fraenkel axiomatisation, so they all fulfil the foundational roles equally well. Consequently, UNIFY is not enough to spot difference between them, but we can use MAXIMIZE to argue *which* set theory is the better axiomatisation. Does this mean that our quest for a method to compare foundational frameworks is complete? Sadly, this is not the case. There are two problems with this method.

First, there is a clear tension between UNIFY and MAXIMIZE. A very powerful theory (according to MAXIMIZE) might be not foundational enough, while a foundational framework may lack proving power. For example, consider the differences between ZFC + $V = L$

⁸For an introduction to category theory see Leinster (2014) and Awodey (2010).

⁹Maddy (2017) gives some arguments against HoTT/UV that are similar to the ones against category theory. See also Maddy (2019) where the issue is treated in more detail.

and $ZF + AD$. The former is strongly foundational: the constructible hierarchy L gives us a very nice universe, where everything we can build is built from something that we have already build. This has the consequences that everything is definable and that a lot of open and independent questions are settled (e.g. CH is true). However, it lacks all large cardinals from measurable upwards, thus making it much weaker than simple ZFC . $ZF + AD$ is equally unacceptable as a foundation. It is much stronger than $ZFC + V = L$: it has all large cardinals (even the choiceless ones!) and it can prove results about the reals. However, the fact that AD is incompatible with AC makes $ZF + AD$ less preferable as a foundation of mathematics: without AC we are losing very nice results in real analysis and other areas of mathematics. Adding new proof methods and mathematical objects doesn't mean anything if in the meantime we are losing some proof method already accepted and widely used.

Second, even $MAXIMIZE$ has its limits. The main one is the arguments that Maddy (1997) calls “devil’s advocate arguments”. Such an argument argues that $ZFC + V = L$ maximises over $ZFC + LCAs$ by constructing an isomorphism type that the former theory can witness, while the latter theory cannot.¹⁰ Moreover, there is also the problem about comparing equally strong (or weak) theories. For example, consider the theories $ZFC + V = Ultimate - L$ and $ZFC + PFA$. Using Maddy’s argument. we could argue that one theory proves the existence of a certain object that the other theory cannot prove. But then we could easily reverse the argument, and there is not an easy way to know which one is the “devil’s advocate argument” here. What we have is two different isomorphism types and no way of arguing which one is the restrictive one.

Consequently, we need something else to settle the question, since we all the methods presented up to this point we get “stuck” in an never-ending argument in which both sides can appeal to exactly the same arguments. We need a new principle that can be added to naturalism ($UNIFY$ and $MAXIMIZE$). Such a principle can then be used to compare possible foundations of mathematics that, if investigated only under the lenses of $UNIFY$ and $MAXIMIZE$, would end up to be equivalent. I argue that such a principle is *Neutrality* over set-theoretic practice, and it’s the main topic of the next sections.

3 Neutrality as a solution to Gödel’s Programme

Lets take stock. I argued that with intrinsic and extrinsic justification, and Maddy’s naturalist maxims, we cannot make any progress on the problem of axiom justification. Most collection of axioms could be justified, or argued against, on naturalist grounds.¹¹ So while we can use (as, for example, Maddy (2017) does) naturalism ($UNIFY$ and $MAXIMIZE$) to argue, in broad strokes, that a particular revisionist foundations of mathematics is worse than classical set theory, things get more complicated when trying to use the

¹⁰See also Löwe (2001), Löwe (2003), Incurvati and Löwe (2016), and Hamkins (2014) for some more arguments in favour of $V = L$.

¹¹This is mainly because philosophers are a polemic group of people, and it might be possible to find philosophical arguments for any collection of axioms. Mathematicians tend to be more measured in this regard, and for them it is true that some collection of axioms are disregarded as “not interesting”. Thus the weakening of the claim from “any” to “most”.

same principles to argue for a *particular* axiomatisation of set theory. In each case, we end up with two opposing arguments that appeal to the same principles and evidence: both $ZFC + V = Ultimate - L$ and $ZFC + PFA$ claim to be intrinsically justified, with good mathematical consequences, good foundations of mathematics, and that the opposing candidate is restrictive.

In what follows, following the considerations of Saharon Shelah on the future of set theory (in Shelah (2002)) and Maddy’s naturalism as just explained, I argue that, since the main philosophical burden of a foundational framework consists in showing that it fulfils its *foundational* purposes *and* that it is not disruptive of set-theoretic practice, its *neutrality* regarding set-theoretic practice could be a new guiding principle to be added to our axiom selection and justification method. The hope is that, when approaching two set theories that seems to be equivalent under any other method (i.e. intrinsic and extrinsic justification, Maddy’s naturalism), comparing which one is more “neutral” (i.e. not disruptive) of set-theoretic practice should allow us to make progress in the debate. However, to anticipate here what I will argue in the next sections, this is not possible, and also “neutrality” ends up with the same problems of the other methodologies. In the last section I will point out a possible way out of the impasse.

3.1 Neutrality towards set-theoretic practice

Shelah (2002)’s reflections on the future of set-theoretic practice are a good starting point to tackle the question of neutrality. In that paper, Shelah provides insights on what he considers the main reasons he is interested in some parts of set-theoretic practice, as opposed to others, and how set theory, in his view, will develop in the following years. He tries to answer the following questions:

1. what’s the main *source of interest* for set-theoretic practice;
2. which framework is the best one to carry out set-theoretic practice;
3. what kind of interest lies behind set-theoretic practice;
4. which size are the most interesting sets.

Ideally, a neutral foundations of mathematics should respect all possible sources of interests, encompass all possible frameworks, accept mathematicians no matter their approach, and finally allow for investigating any possible set of any possible size. It is consequently natural to add the following formulation of NEUTRALITY to our justification methods:

Principle 3 (NEUTRALITY). *A foundational theory T should be as neutral as possible towards set-theoretic practice. More precisely:*

1. *it shouldn’t restrict the kinds of objects we can define, or proof methods we can use;*
2. *it shouldn’t restrict the size of our universe (or objects) without a good reason to do so (e.g. to avoid paradoxes);*

3. *it shouldn't restrict the proof methods available (e.g. a constructivist/intuitionist foundation would be less neutral than a classical one). This is crucial for proof adjudication.*

Intuitively, this means that a theory T should not give us a biased perspective towards set-theoretic practice.

The NEUTRALITY principle just proposed is compatible with Shelah considerations on the practice of set theory. Clearly, if we put any kind of restriction on the objects or their size that are available in the foundation of mathematics, the set-theoretic practice will be equally restricted in some respect. For example, suppose that the theory $ZFC + V = L$ is chosen as our foundation of mathematics. We already discussed how this choice might be problematic, and according to the NEUTRALITY principle (and Shelah's consideration) this is even more clear: we would be restricting our practice to constructible universes, thus losing all mathematics that presupposes the existence of (at least) a measurable cardinal. In the same vein, also the restriction of proof methods to only intuitionist methods (so our foundation would be something like Intuitionistic Set Theory IZF or Constructive Set Theory CZF) would make our investigation of large cardinals problematic and restricted only on "small" large cardinals (see for example Myhill (1973)).

An obvious objection to the addition of the NEUTRALITY principle is that it might seem just a rehashed version of Maddy's naturalism and extrinsic justification. While there are some important similarities, there are also crucial differences that warrant its addition in our justification and comparison methods in the foundations of mathematics.

First of all, let's consider the problem of proof adjudication. One of the most important uses of a foundations of mathematics is helping us deciding whether a proof is actually a proof or not. This is acknowledged by both Maddy (2017) and Shelah (2002). The former considers the foundational role of proving a Shared Standard for proofs central in any discussion around which theory is the better foundations of mathematics:

In this foundational role, then, formal derivation in set theory serves as a Shared Standard of what counts as a proof. [...] formal derivation turns out to be a good mathematical model for the scope of human proving activity [...]. [Maddy (2017, p.15)]

In the same vein, Shelah (2002) contends that something is *definitely* proved if and only if it is proved in ZFC :

[...] all of us are actually proving theorems in ZFC . [...] proving a theorem means proving it in ZFC , [...]. [Shelah (2002, p.3)]

Consequently, in principle, one of the things that our methods of justification of a foundational theory should check is which of our candidate theories manages to discriminate better between proofs and non-proofs. As a quick example, consider again the comparison between constructive set theory (either axiomatised as IZF or CFZ) and classical set theory (ZFC). When deciding whether a particular proof is indeed a proof, we could check if there are (inevitable) uses of the law of the excluded middle or one of its equivalents: if this is the case, then if our foundational framework is constructive then we are

not looking at an acceptable proof. And this is the main idea behind Maddy’s Shared Standard: we need a common framework in which we can check whether theorems are actually proved or not.

The question now arises naturally: is the NEUTRALITY principle useful enough in providing a foundational theory with enough “bite” to adjudicate between proofs and non-proofs? If taken alone, NEUTRALITY is not enough. Simply put, it is a too permissive principle that would allow foundational theories that cannot serve as a good Shared Standard for proofs. However, NEUTRALITY is not supposed to be taken alone: it is an *addition* to extrinsic justification and Maddy’s naturalism. Consequently, we still get to compare possible foundational theory by their ability to provide a Shared Standard, since we still have the foundationality roles from Maddy (2017). NEUTRALITY is not going to replace them, it is only going to support them in the cases in which Maddy’s naturalism is not enough. And the same goes for the other foundationality roles: we can still compare theories using Maddy’s naturalism and only use NEUTRALITY to complement that. But now the question becomes: what does NEUTRALITY actually add to naturalism?

It might be the case that if we add this principle to our array of justification methods (so to intrinsic and extrinsic justification, and to Maddy’s maxims) we might get a better and fuller picture when comparing different (set-theoretic) foundations of mathematics. Consider the example from last section. We cannot compare $ZFC + V = Ultimate - L$ and $ZFC + PFA$ since they are both natural extensions of ZFC. In particular, $V = Ultimate - L$ claims that the universe has a certain nature, while PFA gives us an answer to which kinds of forcing are possible. The problem in approaching the question of which one is the better extensions lies in the fact that they both have nice consequences (e.g. $V = HOD$ ¹² and no absolute undecidability for $V = Ultimate - L$, solution to CH for PFA), they are both based on ZFC and thus foundational (UNIFY features are all satisfied), and they can both prove the existence of isomorphism types that the other cannot. So we cannot prefer neither of them on the basis of intrinsic justification, extrinsic justification, or Maddy’s naturalism.

The idea is that at this point the NEUTRALITY principle comes into play. If the principle has been defined correctly, we can compare two set theories in a much more precise manner. An ideal solution would be a situation where two set theories T and T' seem the same through the lenses of all the other justificatory methodologies (naturalism, extrinsic justification, etc.), but one of them is more neutral than the other. For example, consider the case of $ZFC + PFA$ and $ZFC + PD$. At first glance, the choice seems easy: PFA implies PD, so since with $ZFC + PFA$ we can get everything that $ZFC + PD$ provides, we should simply chose PFA. However, things are not that simple, and the debate for and against forcing axioms like PFA is still quite heated. If we try to compare the two axioms foundationality, we go nowhere. Both are set theories extending ZFC, so we can expect them to easily satisfy all the foundationality roles from Maddy (2017). However, if we try to approach the question using MAXIMIZE, we can come up with a clear answer! Since PFA implies PD, the latter will be restrictive over PFA, and thus the former should

¹²HOD is the class of all *hereditarily ordinal definable* sets, i.e. the sets that are ordinal definable and moreover all the elements of their transitive closure are also ordinal definable.

be preferred over it. With PFA we get the negation of CH, while giving us the possibility of doing proper forcing. Moreover, since it implies PD, we also get that sets have very nice properties (they are all Lebesgue measurable, perfect, and Baire), while still being consistent with AC (as opposed to full determinacy). This seems to settle the matter, but instead this is not the case, and forcing axioms are still not considered the a worthy extension of ZFC.¹³ Such a situation is clearly problematic for the naturalist: following the maxims of UNIFY and MAXIMIZE, we get a precise answer when comparing two axioms, but the wide majority of practitioners seems to ignore it.

Let's instead consider the problem through the lenses of NEUTRALITY. We notice that PFA imposes a "restriction" on the size of the continuum, since it implies the negation of CH and that $2^{\aleph_0} = \aleph_2$. On the other hand, ZFC + PD doesn't impose a similar restriction. Consequently, following the NEUTRALITY principle, we reach the conclusion that ZFC + PD is more neutral than ZFC + PFA as a foundations of mathematics, and for this reason preferable, even though PFA implies PD and would be naturalistically preferable. Such a line of argumentation seems promising enough. We start by using Maddy's naturalist maxims to determine two possible candidate axioms, and then, if they seem equivalent under the lenses of Maddy's MAXIMIZE (and the other methodologies in general), we try to see which one avoids restricting set-theoretic practice too much, according to NEUTRALITY.

As I will argue in the next section, this is still not enough. To anticipate: the NEUTRALITY principle ends up having to deal with the same problems that the other methodologies have. Consequently, no matter the promise it shows in comparing theories that otherwise seem equivalent, it still foster good looking arguments for both sides of the debate.

3.2 Some problems

The proposed NEUTRALITY Principle seems promising. When investigating the debate between determinacy axioms and forcing axioms, for example, it is able to track down a plausible reason why forcing axioms aren't overwhelming preferred by set theorists, even though from the naturalist perspective they should be better off.

Lets consider another example. Consider the debate around the axiom $V = \text{Ultimate} - L$ and forcing axioms like Martin's Maximum (MM). From the perspective of consistency strength, MM sits very high in the hierarchy, between supercompact cardinals and Woodin's cardinal (but the precise consistency strength is still not known, see Foreman, Magidor, and Shelah (1988) and Shelah (1987)). On the other hand, it is conjectured that $V = \text{Ultimate} - L$ has an even higher consistency strength, at the level of I2 or I3 (see McCallum (2018), still unpublished). So in terms of consistency strength, we should prefer $V = \text{Ultimate} - L$ over MM. However, from the perspective of MAXIMIZE, the situation is different: as Schatz (2019) shows, $V = \text{Ultimate} - L$ is *restrictive* over MM, thus implying that, from the naturalist perspective, MM is preferable over $V = \text{Ultimate} - L$.

¹³See for example Steel (2024), in which the preference of other axioms over forcing axioms is made explicit.

It seems that in this case there is a similar impasse like in the case of $ZFC + PFA$ and $ZFC + PD$.

The application of the NEUTRALITY principle should then follow the same strategy than in the case we investigated in the previous section: either $V = Ultimate - L$ is more neutral than MM , or the other way around. However, such a line of argumentation fails in this case, since they seem equal in this respect. $ZFC + Ultimate - L$ restricts the size of the continuum to \aleph_1 , while MM to \aleph_2 . However, MM gives us more powerful proof methods (i.e. different kinds of forcing), while $V = Ultimate - L$ gives us the possibility of having both very large cardinals in a inner model. Consequently, it seems that they are more neutral in some respects, while being more restrictive in some other.

The kind of impasse that we get in this case with the NEUTRALITY principle is of the same kind that we already encountered with intrinsic and extrinsic justification, as well as with Maddy's maxims: both sides of the debate can to appeal to the same arguments and the same principles to support their axiom, and no progress can be made. So, where did we go wrong with NEUTRALITY? Is its definition wrong (and thus completely redundant over Maddy's naturalism), or are we applying in the wrong way? Considering how similar the NEUTRALITY principle seems compared to Maddy's maxims and extrinsic justification, it would be difficult to argue that it is an outright wrong principle. This leaves the possibility that the way I described its application is not the right way. I claim that it is the latter reason, and furthermore that this is closely related to the problems highlighted about the other methodologies in the previous sections.

The problem here is that all the above solutions (intrinsic and extrinsic justification, Maddy's maxims, NEUTRALITY) assume universist attitude towards foundational questions and set-theoretic practice (since based on Gödel's Programme). *Universism* is the position that there is a single universe of set theory, the cumulative hierarchy of sets V . According to universism, every mathematical proposition that can be formulated in the language of set theory is either true or false in the true universe of sets, thus eliminating any possibility of indeterminacy. This is an (implicit) assumption behind Gödel's Programme and all the justification methodologies described in the previous section. The goal of Gödel's Programme (as we saw at the start of this paper) is to find *the* best justified axiom A such that $ZFC + A$ settles the independent statements. But, ly, all the subsequent debate has been aimed at producing arguments in favour or against particular axioms (like we saw in all the examples of this paper). In each of those cases, the expectation is that one of the axioms will end up being better justified than the other, thus become the adopted extension of ZFC while the other axiom will be abandoned.

For example, if we adopt intrinsic justification as our methodology, we will say that axiom A is directly implied by our concept of set, while axiom B is not and thus it doesn't produce the "right" set theory for us. For extrinsic justification, this instead means arguing that the consequences of A are better then the ones of B , so much so that we "don't care" of missing them if we adopt $ZFC + A$. Similar arguments are used in the case of Maddy's maxims: $ZFC + A$ is foundational (while $ZFC + B$ is not), or $ZFC + B$ is restrictive over $ZFC + A$. The same can be said of of NEUTRALITY: one theory is neutral, thus better and ought to be adopted, while the other is not, and consequently is better to avoid it.

The problem with this attitude is that the choice between extensions of ZFC is made difficult by the fact that there are mutually inconsistent and exclusive extensions (e.g. $ZFC + V = Ultimate - L$ and $ZFC + MM$, or $ZF + AD$ and ZFC) that are, nevertheless, mathematically equally interesting (this is recognised, for example, by both Hamkins (2012) and Shelah (2002), even though the latter is somewhat more ambiguous since it still gives the precedence to ZFC, see the quote in the previous section). We saw in the previous sections that impasses in the comparison between such axioms are highly problematic, and all the methodologies presented are prone to them. Any argument that seems to favour one particular theory will find objections from the excluded theories.

Summarising, if we try to apply the new NEUTRALITY principle just like we applied all the other methodologies, we run into the same problems. In the next section I will propose a different interpretation of the NEUTRALITY principle, closely related to pluralism instead of universalism and Gödel's Programme that should give us a new way of comparing set theories that it's not prone to the the problems highlighted so far (or so I claim).

4 Neutrality and set-theoretic pluralism

In the last section I argued that NEUTRALITY faces the same problems that beset all the other methodologies. Moreover, I argued that the problem lies in the universalist attitude towards set-theoretic practice that's intrinsic to our approach to Gödel's Programme. Intrinsic and extrinsic justification, Maddy's maxims, and NEUTRALITY (as applied in the last section) try to solve the problem of axiom justification inside the context of Gödel's Programme, by comparing different axioms and trying to come up with the "more justified" one, that will then be added to ZFC and settle our independent statements. However, as I argued, this presupposes a flawed universalist perspective that cannot give us definite answers on which axiom to prefer.

An alternative view to universalism is *mathematical pluralism*, according to which many different mathematical objects coexist – be they mutually compatible or not. The set-theoretic characterisation of pluralism is *Multiversism*, which asserts that there is no single set-theoretic universe that embodies the correct description of the set concept; instead, each model of every consistent set theory can potentially serve as a legitimate universe within a set-theoretic multiverse. Various mathematical characterisations of multiversism exist. Hamkins's multiverse Hamkins, 2012 is certainly the most radical form of pluralism. He considers all universes equally legitimate and ontologically *on a par*. They all coexist without any absolute hierarchy, even if pragmatic reasons may favor one over another. Steel's set-generic multiverse Steel, 2014 consists of all set-generic extensions of a core universe.¹⁴ Väänänen's parallel multiverse Väänänen, 2014 comprises all cumulative hierarchies defined by different power set operations, and utilises dependence logic to define and access these universes. Friedman's Hyperuniverse Antos et al., 2018

¹⁴There's also Woodin's set-generic multiverse Woodin, 2011, but he uses it to argue against multiversism and it ultimately collapses back to universalism. For Steel's multiverse, see also Maddy and Meadows, 2020; Bagaria and Ternullo, 2020.

includes all countable transitive models of V using V -logic. Despite their mathematical differences, these approaches share the common feature of rejecting the existence of only a single ontologically privileged universe.

All the methodologies presented don't sit well with such a pluralist approach, since Gödel's Programme tend to prefer a universalist perspective on the problem of axiom selection and justification (as we saw). If we accept the possibility of more than one legitimate concept of set, there is no point of arguing about the intrinsic justification of a particular axiom, since each axiom simply instantiates a different concept of set (all equally legitimate). Moreover, if we can have several different automatisations and their models in a single multiverse, then there is also no point in comparing the extrinsic justification of those axioms, since we don't actually need to choose one. Similar arguments can be made in the case of Maddy's MAXIMIZE: while the investigation of the relative strength of set theories remains a worthy research field, in a pluralist framework there is no need to pursue it with the final goal of choosing the least restrictive theory, since in the multiverse we can already find all the possible models of the theories involved in the comparison. In the same vein, UNIFY loses its purposes of finding a foundational extension of ZFC. Finally, while NEUTRALITY is not, per se, able to make any significant contribution to the debate on the execution of Gödel's programme, it can, however, be seen as a priority naturalistic maxim (since it is, in a way or the other, derivable from all other criteria) which encourages one to adopt a pluralist point of view.

Why can NEUTRALITY be seen as supporting a pluralist approach to the foundations of mathematics? Indeed, it might be argued that any strong commitment to a single foundation would necessarily be less neutral than no choice at all. First of all, the choice of a single foundation restricts the kinds of objects we can work with. For example, consider the case of choosing which set theory is the best (better) foundation. The consensus is that ZFC is our foundation of mathematics. However, with this choice, we are restricting ourselves to operate without determined infinite games (that would instead be available in $ZF + AD$), or non-trivial universe embeddings (available in $ZF +$ "there exists a Reinhardt cardinal"). On the other hand, if we only commit to ZF as our foundation, we don't face this restriction. For a more general example, consider the choice between set theory or category theory as our foundation of mathematics (see Maddy (2017) for a summary of the possible arguments in favour or against each candidate). If we choose set theory as our foundation we are forced to see categories and morphisms between them as set-theoretic objects, and that could be unwieldy (see for example Barr et al. (1969) and Osiris (1974)). On the other hand, if we choose category theory as the foundation of mathematics then we miss some of the foundational roles (as argued by Maddy (2017)). A similar argument can be made for the size of the mathematical universe and the proof methods available. In the case of the universe size, committing to $ZFC + MM$ as our foundation will commit ourselves to a set-theoretic universe in which the continuum has size \aleph_2 (since MM implies the falsity of CH). The same goes if we accept $V = Ultimate - L$, in that case CH is true and the continuum invariably equal to \aleph_1 (this style of argument can be found in Hamkins (2012)). What about all the other possibilities? Moreover, each different choice of foundations comes with a different variety of proof methods available. This is very easy to see when comparing foundations based

on different background logics: for example, intuitionistic set theory (either axiomatised as IZF or CZF) will ban all the proofs using the law of the excluded middle and the the proofs by contradiction, with cascading effects on the kind of mathematics we can do. However this is true also in the more subtle cases of choosing between forcing axioms and constructibility axioms like $V = \text{Ultimate} - L$. Clearly, any choice is a restriction.

A pluralist approach to the foundations of mathematics, like the one developed by Hamkins (2012), would allow us to chose not one single theory (and by that losing on all the benefits of the other theories), but several of them, and integrate the often incompatible constructions (such as $ZFC + CH$ and $ZFC + \neg CH$, together with their respective models) into a single set-theoretic multiverse. Following this argument, the theory of a set-theoretic multiverse, for example Steel’s set-generic multiverse (see Steel (2014)), will count as a better foundations than a universalist, single theory like $ZFC(+LCs)$. It is then possible possible to use the NEUTRALITY principle to compare the different set-theoretic multiverses, but this is the topic for another paper.

To sum up, the NEUTRALITY allows us a novel perspective on the axiom selection problem. If we stick to the universalist perspective of Gödel’s Programme, NEUTRALITY fall into the same problems of the other methodologies, since we also try to get a *maximal* theory (like the other justification methods do). On the other hand, NEUTRALITY seems to provide arguments in favour a *minimal* solution to Gödel’s Programme, a solution that supports a pluralist view of the foundations of mathematics. In other words, for the NEUTRALITY principle the most minimal set theory, that allows us to prove *the bare minimum*, while leaving us all the possibilities of expanding it, is the best possible foundation of mathematics. Of course, the problem here lies in defining what this “bare minimum” actually is. However, this aligns with the research goals of reverse mathematics, i.e. the project of determining which axioms are needed to prove which theorems (see for example Simpson (2017)).

For these reasons, I contend that the NEUTRALITY principle is a worthy addition to the justification methods normally used in the foundations of mathematics. Moreover, by allowing us to compare even pluralist frameworks, the NEUTRALITY principle ends up being more useful than the usual justification methods.

5 Concluding remarks and outlook

In conclusion, in this paper I argued that all the methods introduced to justify new axioms to achieve Gödel’s Programme have problems when applied to cases in which the axioms compared are mutually incompatible but mathematically equally interesting. This is because they all assume a universalist perspective on the solution to Gödel’s Programme, and thus they don’t help us in the choice of a foundational framework.

On the other hand, the NEUTRALITY Principle, that I introduced in this paper, supports a pluralist solution to Gödel’s Programme. Such a solution allows us to make some progress in the debate, and consider a more wide array of set theories (and multiverses) as properly foundational. For these reasons, I argued that its addition to the other methods is a worthy one, and can help in making progress in the foundational debate of axiom

selection and justification.

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