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Introduction

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More than a Decade in the Set-Theoretic Multiverse

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1 Introduction

The importance of set theory can hardly be overestimated: from its first development by Georg Cantor and Ernst Zermelo, to the most recent results, set theory has always been seen as providing a foundation for mathematics. And this is for an important reason: it is possible to model and develop all of mathematics in set theory. Set theory is usually identified with its standard axiomatisation, Zermelo-Fraenkel with Choice (ZFC), which is often taken to formally articulate our intuitive conception of sets and membership. The seminal work of Zermelo first suggested a "cumulative" conception of sets, according to which sets are constructed in stages and form a v-shaped structure, the universe of set theory V. The cumulative hierarchy V was defined in such a way that encompassed our intuitive notion of sets and membership, while avoiding to fall pray to the set-theoretic paradoxes that plagued the early development of set theory. However, from the sixties onward it became clear that some propositions in the language of set theory cannot be proved from the canonical axioms of ZFC – the Continuum Hypothesis (CH), a central set-theoretic claim, being a case point.[1] The independence[2] of CH from ZFC, i.e. the fact that ZFC doesn't prove neither CH nor $\neg CH$, strongly suggested that the usual set theory was inadequate as it stood. Whence the quest for finding an adequate extension of ZFC, usually referred to as *Gödel's Programme*, started: new axioms were proposed, but none of these new

1 The Continuum Hypothesis is concerned with the size of infinite sets: according to this hypothesis, there are no sets with cardinality strictly bigger than the set of natural numbers but strictly smaller than the one of real numbers.
2 For a classic primer on independence proofs and models of set theory, see Kunen (2011).

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axiomatisations were stably accepted as “the” axiomatisation of set theory. The reason is that several such new axioms are mutually exclusive: choosing one implies that all the results proved under an incompatible axiom can no longer be proven. This led to competing set theories, and to the necessity to develop a formal methodology to compare the axiom candidates. At present, none of these theories is accepted as the “new” set theory.

The independence phenomenon is both pervasive and inevitable, something which has led some set-theorists and philosophers to support a *pluralist* conception of sets. In recent years, such a conception has emerged and progressively gained prominence in the debate on the foundations of mathematics. More precisely, the independence phenomenon has manifested itself through a proliferation of models of set theory. To wit, in order to show that statements such as CH are independent of set theory, models have been constructed which settle the truth of these statements in *multiple, mutually incompatible* ways. According to pluralism, despite the relatively early emergence of a proper class-sized[3] structure $(V, \in)$ encompassing all sets, which may naturally be seen as playing the role of “single universe”,[4] set theory is not the theory of a *single* set-theoretic structure, but rather of *multiple* set-theoretic structures. Such a pluralist view is then mathematically instantiated by the *set-theoretic multiverse*.[5] Opposed to this view is *universism*, the thesis that there exists only one universe of set theory, that instantiates the “One True Set Theory”. Adding new axioms to ZFC will eventually produce an axiomatisation that settles all the open (and independent) questions, thus pinning down which one is the “true” set-theoretic universe, the canonical and intended model of this new axiomatization of set theory.[6]

To anticipate (making use of technical notions that will be introduced in due course), there are several kinds of set-theoretic models, but the following ones are the most relevant for the multiversist’s purposes: set-models in V (e.g., countable transitive models), inner models, and outer models. Inner models are transitive subclasses $(M, \in)$ of V which arise from somehow restricting the Power-Set operation (Gödel’s constructible universe L here is the standard example). Outer models are models obtained through *forcing*. On the standard interpretation of forcing, one starts with a countable ground model $\mathscr{M}$ and extends it to a model $M[G]$ which

3 A proper class is a collection of sets that is not itself a set, either because is too big or because it is defined by a paradoxical property. For a discussion of classes see Shoenfield (1973).

4 For further historical details, see e.g. Hallett (1984), A. Kanamori (1996), Ferreirós (2010).

5 For a general account, see Antos et al. (2015).

6 From now on, as it is usually done in the literature, I will use “model” and “universe” interchangeably.

contains a generic filter G[7] over a poset[8] $\mathbb{P} \in M$. By contrast, on a pluralist interpretation, outer models are seen as proper classes[9] $V[G]$, which should, in turn, be seen as extensions of a *ground* universe. By "outer model" here I mean a model obtained through *set-forcing, class-forcing, hyperclass-forcing* (i.e. forcing applied to sets, classes with set conditions and classes with class conditions respectively) and, in general, any model-theoretic technique able to produce *width extensions* of V (intuitively, extensions resulting from adding "new" subsets to V). Since the claim that it is possible to extend V is at least philosophically problematic,[10] a face value interpretation of $V[G]$ is formally barred. However, there are ways to make sense of such an interpretation of forcing.[11]

These set-theoretic models involve different possible attitudes towards the extendibility of V. *Actualism* is the view that V is a fully determinate, inextensible object, whereas *potentialism* is the view that either the height (the length of the sequence of the ordinals) or the width (the extent of the Power-Set operation) of V are open-ended (or both), and thus extendable.[12] Radical actualism may be seen as a different way of expressing the universist view, since, clearly, universists must hold that there is a fully determinate entity that is referred to by the axioms. On the other hand, ontological pluralism *prima facie* appears to be more in line with a radical potentialist understanding of set-theoretic reality, although this is not inevitable.[13]

A pluralistic conception of sets can be mathematically characterised in different ways, which give rise to different set-theoretic multiverses. At one extreme we have Hamkins' *broad* (or radical) multiverse, in which all universes are on a par: all models of any (consistent) set theory are legitimate – all provide an acceptable interpretation of the membership relation. At the other end, we have the *set-generic* multiverse, in which all universes are produced by a single, "core" universe and they all witness the same basic theory.[14] One could also define the universes of the multiverse using a strong logic, like, for example, Friedman's Hyperuniverse

7 A generic filter is an object used in the application of forcing, and can be, intuitively, imagined like a "missing subset" of a set, with a number of special features that change the original model.

8 A poset is a partially ordered set, i.e. a set ordered by a binary relation.

9 A class is a set-theoretic object that cannot be a set, usually because is too big or general.

10 If V is taken to be the all encompassing universe of sets, then all sets should be already in V, and thus talking about "missing" subsets of V is problematic.

11 For instance, through the Boolean-valued model approach, whereby $V[G]$ is taken to be equivalent to $V^{\mathbb{B}}$, for some complete Boolean algebra $\mathbb{B}$. See Bell (2005).

12 For further details on the actualist/potentialist divide, see e.g. Koellner (2009a), Ø. Linnebo (2013), and Antos et al. (2015).

13 For an *actualist* interpretation of "extensions of V", see Antos et al. (2021).

14 For a detailed account of the broad multiverse, see J. Hamkins (2012); for the set generic multiverse, see Steel (2014). For some details on a possible classification of multiverse conceptions, see Koellner (2009b).

(see S.-D. Friedman 2016), or define them as different cumulative hierarchies arising from several power set relations with varying definitions (see Väänänen 2014). Thus, rejecting the standard universist conception of sets, or any conception of sets according to which there is a single universe of sets, by no means settles the question concerning the nature and justification of the multiverse.

In the rest of this introduction I will introduce some background notions and context behind this special issue. I begin by presenting the classic set theory ZFC (Section 2). I then describe its axioms (Section 2.1) and the corresponding set-theoretic universe V generated from them (Section 2.2). In my next step, I discuss independence results and the main reason behind the need of extending ZFC (Section 2.3). Then, I will first explain the first proposals of extending ZFC (Section 3). After that, I move on to introduce the two main side of the recent debate in the foundations of mathematics: universism (Section 4) and the multiverse conception of sets (Section 5). Finally, an outline of the contributions included in this special issue (Section 6) concludes the introduction.

2 From ZFC to the Multiverse Conception of Sets

In this section, I outline the success story that is ZFC and classical set theory. First of all, I introduce ZFC, its axioms (Section 2.1), and its underlying conception of sets, viz. the *iterative conception* (Section 2.2). Section 2.3 then discusses the main problems afflicting ZFC and its underlying conception of sets: the so-called independence propositions, i.e. propositions that are logically independent from ZFC, such as the Continuum Hypothesis.

2.1 Zermelo-Fraenkel with Choice: The Basic Set Theory

Set theory is a mathematical theory devoted to the study of *sets* from a formal point of view. It is the main research field for anyone interested in the foundations of mathematics, that is, for anyone interested in the logical and philosophical foundation of mathematics.[15]

Historically, set theory was first developed by Cantor and Richard Dedekind in the late 19th century, while trying to formalise the concepts of *set of points* and *set of*

15 It is not however the only possibility. Category theory (see Simmons 2011) and Univalent Foundations (see Voevodsky 2011) are the two main alternatives to Set Theory.

reals.[16] The linguistic and formal conventions (e.g. the quantifiers) developed in the same period by Gottlob Frege,[17] and the notation and the syntax developed by Giuseppe Peano,[18] were soon incorporated in the theory, thus making possible a first axiomatization (by Zermelo (1908), further developed by Abraham Fraenkel (1922)). This first axiomatic system (ZF) was later enhanced by Paul Bernays, Thoralf Skolem, Johann von Neumann and Gödel, with the addition, among other things, of the Axiom of Choice (AC), thus yielding the standard set theory Zermelo-Fraenkel with Choice, ZFC.[19] After this first period, characterised by an optimistic development of what was considered *the* foundation of mathematics, Gödel (1931) first (with the incompleteness theorems) and Cohen (1964) after him (with the proof that the Continuum Hypothesis, CH, is not provable in ZFC) took away much of the optimism that accompanied the initial development of ZFC: set theory, as axiomatised by ZFC, was clearly not enough to uniquely pin down *the* structure of sets (as the Peano Axioms are thought to be pinning down the structure of the natural numbers). More recently, the further development of forcing,[20] and the investigation of inner models[21] and large cardinals[22] gave new strength to ZFC. Consequently, at the beginning of the 21st century, there was the widespread hope that set theory, as axiomatised by ZFC, can hope again to become "the" foundation for mathematics.[23]

The main point of interest of set theory is the fact that, from a very simple collection of axioms, it is possible to formalise the whole mathematics, from abstract algebra to chaos theory. With the axioms of ZFC it is possible to model all of the

16 Dedekind's contributions to the rise of modern set theory are mainly in Dedekind (1965a, 1965b). Cantor's contributions are so vast that it is not possible to list them all, but see G. Cantor (1885) for the article in which he introduces set-theoretic techniques for defining set of points, while two very important articles are G. Cantor (1895) and G. Cantor (1897).

17 See Frege et al. (1879).

18 See for example Peano (1908).

19 A translation of all the most important contributions to the development of Set Theory can be found in Van Heijenoort (2002). A good historical account of this first development of modern Set Theory can be found in Ferreirós (2010).

20 See for example W. H. Woodin (1999).

21 A good description of recent development in inner model theory is Steel (2009).

22 Large cardinals are transfinite cardinal numbers with some additional properties, for example being "inaccessible" (that is not obtainable from smaller cardinals using the usual operations) or "supercompact" (that is there exists an embedding $j\colon V \to M$, where M is an inner model with critical point κ, $j(\kappa) > \lambda$ and M contains all its λ-sequences). A *large cardinal hypothesis* is an axiom that states the existence of such large cardinals. See Section 3 and A. Kanamori (2003) for more details.

23 More precisely, this is considering ZFC (+LCs) compared against other alternative set theories, like ZF + AD or ZFA. Again, ignoring Category Theory and Univalent Foundations, the actual debate around the proper foundations of mathematics is more complex.

known mathematics. Indeed, in a similar way to scientific reductionism, according to which every science is in the end reducible to physics[24] ZFC is usually taken to be a partial, but accurate, description of a universe of sets that goes from the empty set to infinite sets: the cumulative hierarchy.

2.2 The Universe of ZFC: The Cumulative Hierarchy

The intended universe of set theory (the universe of all sets) is usually referred to as the *cumulative hierarchy*, or von Neumann's hierarchy. So, how is the cumulative hierarchy built? First, we consider only one set: the empty set. This is the first level of the hierarchy.[25] Then, every new finite level is constructed using the power-set operation to form a new level with all the subsets of all the sets of the previous level. When we reach a limit level, we take the union of all the previous no-limit levels. In this way, we can prove for any set that it is in a level of the hierarchy. Thus, the universe of sets V consists of (ordinal-indexed) levels, recursively defined as follows (also, cfr. Figure 1):

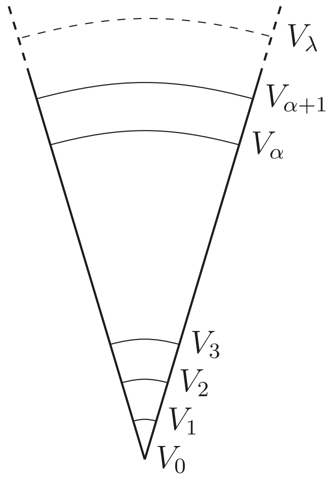

Figure 1: The cumulative hierarchy.

24 Whether this is actually the case or not is still debated, as well as whether it makes sense to reduce everything to physics. See Nagel (2007) for an account of the debate.
25 A different approach uses *Urelementen*, that are not sets but can be members than sets. The two approaches are essentially equivalent, see for example Löwe (2006).

$$V_0 = \varnothing;$$
$$V_{\alpha+1} = \mathscr{P}(V_\alpha);$$
$$V_\lambda = \bigcup V_\alpha \text{ (for all } \alpha < \lambda \text{, where } \lambda \text{ is a limit ordinal).}$$

The construction of V seems to be very aptly reflected (and intuitively motivated) by the *iterative conception of sets*, that is, the idea that *all* sets are formed in stages, through iterating the Power-Set and Union operations *as far as possible*.[26]

One natural question at this point concern the nature of V. Given the above construction, it is clear that V contains all sets. But is it a set? If it were a set, it would be the *universal* set. But this is impossible, as we know from Russell's Paradox. Suppose that V is indeed the universal set, the set of all sets. Since the axioms of set theory that I presented in the last section state that no set can be member of itself, we can define the universal set as the set of all sets that are not member of themselves. But we said that V is the *set* of all sets. Thus, it cannot be member of itself. But if this is the case, then it satisfy the condition of being member of V, and thus it is indeed member of itself. But if it is member of itself, then it must not be member of itself, by definition. But this is a contradiction, since we derived that V is member of itself from the premise that it is not, and that it is not member of itself from the premise that it is a member of itself. Consequently, V is not a universal set.

Given this conclusion, there are two different ways to avoid the paradox. The first is to consider a principle of *limitation of size*, according to which V is too big to be a set and is thought to be a *proper class* instead.[27] Classes are collection that can have members, but that cannot be member of anything else (as opposed to sets, that can have member and can be elements of other sets). Thus, we can have the class of all sets without incurring in Russell's Paradox. However, in doing so, we also need to regiment the use of classes. This leads to development of what is known as "class theory". In class theories, we add the possibility of quantifying over proper classes. To avoid the paradoxes and regiment the use of classes, some axioms are also added (mainly some version of a Class Comprehension Axiom). This gives rise to several different axiomatization of class theory. The two main examples of class theories are von Neumann-Bernays-Gödel class theory (NGB) and Morse-Kelley class theory (MK).[28]

26 See e.g. Boolos (1971), Wang (1974), and Parsons (1983).

27 Here the terminology changed during the development of set theory. At first, Cantor referred to "sets" as "classes" (for example in G. Cantor (1879)). After the discovery of the first paradoxes, Cantor referred to the paradoxical classes first as "Inconsistent Multiplicities" (G. Cantor (I, 1887; II, 1888)), then as "Absolute Infinite Multiplicities" as a way to sidestep Cantor's Paradox (Georg Cantor 1897). For a discussion of the first development of set theory, see Giaquinto (2002).

28 For an introduction to MK, see Morse (1986) and Kelley (2017). For NBG, see von Neumann (1928).

Both NBG and MK class theories are very powerful, but neither has been universally accepted by set theorists, for both mathematical and philosophical reasons. In particular, many set theorists are reluctant to accept the ontological commitment to classes. The problem lies in the fact that, as soon as quantification over classes is permitted, through Cantor's diagonalization is possible to show that there is no Universal Class. Thus we are in a similar position as set theory and its antinomies: the supposed Universal Class cannot be a class, since it is too big. So we could assume the existence of "super-classes", and then "super-super-classes", and so on. However, there is little advantage in doing this, since in set theory we never need to quantify over classes.

The other way to avoid Russell's Paradox is to limit the applicability of the Comprehension Schema. The axiom, in its full power, allows us to form a set from any stated property. This is needed to define the universal set as the set of all sets that are not member of themselves. It is possible however to restrict the Comprehension Schema in such a way as to avoid the paradoxes. This is the most common way to approach Russell's Paradox. The restricted version of the Comprehension Schema is the Axiom Schema of Separation: it states that we can form a set of elements that satisfy a given property if and only if they are already elements of another set. In this way, we avoid the possibility of forming a universal set with a given property, since we need to refer every time to another set.

As I mentioned on p. 6, philosophically, there are two possible ways to approach the process of filling up the stages: potentialism and actualism. According to potentialism, the hierarchy is never completed, i.e. we cannot take it to be a completed entity and, for this reason, it never contains *all* sets. On the other hand, according to actualism the whole structure is already completed, i.e. it contains all possible sets. We can mix and match these two interpretations:

– *Radical potentialism*: one can add both subsets (so adding to the *width* of the hierarchy) and ordinals (so one can always increase the *height* of the hierarchy);
– *Radical actualism*: both the width and the height of the hierarchy are fixed;
– *Width potentialism/height actualism*: one can always add new subsets, but not new ordinals (so the height of the hierarchy is fixed, while the width is not);
– *Height potentialism/width actualism*: one can always add new ordinals, but not new subsets (so the width of the hierarchy is fixed, but its height isn't).

The story so far has much to recommend: we have an axiomatic system, ZFC, strong enough to represent virtually every currently accepted mathematical theory and object, with a strong motivation behind it, the cumulative hierarchy. At first glance, then, it seems very natural to think of set theory as the theory of one single entity, the proper class V, whose members are *the* sets. Sadly, however, there is a fly in the ointment: Gödel proved that any system that interprets a modicum of arithmetic (and

ZFC is certainly one such system) is incomplete, in the sense that it does not decide all sentences formulated in its language, such as, for instance, the consistency sentence of ZFC, i.e. a sentence to the effect that ZFC is consistent. Any system satisfying Gödel's minimal conditions, can be seen to be in some sense inadequate. However, it was still believed that set theory was powerful enough to prove any other mathematical result (other than its own consistency). The blow against this belief was struck during the sixties: it was shown that the most important open question in set theory, the Continuum Hypothesis, cannot be solved within ZFC. Even worse, there are many mutually incompatible theories extending ZFC, that either validates CH or its negation (as we will see in more detail in the next section).

Moreover, it is not only such incompleteness problems that undermine the uniqueness of the universe of set theory V. It is also possible to define completely different universes that still satisfy ZFC. One of such different universes is the *constructible universe L*, first introduced by Gödel (1947). The main difference between V and L lies in what sets are we adding at each stage. While in V we are adding *every possible set* at each stage, in L we can only add sets that can be constructed from sets that already present at some previous stage. This gives us a much thinner universe, the first *inner model* of ZFC. Finally, since ZFC is a first-order theory, it is subject to the application of Löwenheim-Skolem Theorem: if a theory T has a model of infinite cardinality,[29] then it has a model of *each* possible cardinality. Consequently, ZFC has not only uncountable models (and this seems intuitive, since it proves that the set of all real numbers is uncountable), but also *countable* models!

Now, while there is a widespread consensus that ZFC is justified by the iterative conception of sets and membership V, the same kind of intuitive justification does not automatically apply to extensions of ZFC, or to L. These different universes, on the contrary, seem to be justified only indirectly (*extrinsically*, to use Gödel's terminology), that is, in view of their consequences.[30]

2.3 The Continuum Hypothesis and Its Independence

The Continuum Hypothesis is probably the main reason behind the development of extensions of ZFC and of the multiverse itself. In this section, I discuss CH and also briefly present the current status of research on the matter.

The CH is a claim about the size of sets: it states that there is no set with more elements than the set of natural numbers $\mathbb{N}$ *and* fewer elements than the set of real numbers $\mathbb{R}$. That is, since the cardinality of $\mathbb{R}$, in symbols $|\mathbb{R}|$, is 2 elevated to the

29 The cardinality of a set is the number of its elements.

30 See e.g. Gödel (1947, 1964).

cardinality of the natural numbers, $2^{|\mathbb{N}|}$, we have that $|\mathbb{R}| > 2^{|\mathbb{N}|}$. And since $|\mathbb{N}| = \aleph_0$, we have that $|\mathbb{R}| = 2^{\aleph_0}$. In other words, CH is the claim that there is no cardinality between $\aleph_0$ (the cardinality of $\mathbb{N}$) and $2^{\aleph_0}$, the cardinality of $\mathbb{R}$. Thus:

$$2^{\aleph_0} = \aleph_1.$$

Despite its simplicity, this statement has important consequences: if CH is false, then we can build a set S with no bijection with the natural numbers, but only with an injection into the real numbers. This means that some element (actually, an infinite amount of them) of S will always be left out and, at the same time, we would have the usual behavior between S and $\mathbb{R}$ (with an infinite amount of real numbers that are not elements of S), thus getting that the cardinality of S is strictly bigger than the one of the natural numbers and strictly less than the one the reals: $|\mathbb{N}| < |S| < |\mathbb{R}|$. And yet, given the iterative process of the cumulative hierarchy we know that, using only the operations used to build up V, there is no way a set such as S could be built in the first place. More generally, CH is fundamental for our understanding of infinite sets and of their arithmetical properties (this is especially true of the so-called Generalized Continuum Hypothesis, GCH, concerning sets of arbitrary size). In particular, the GCH neatly defines the exponentiation between cardinals: since the GCH states that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$, exponentiation becomes a trivial matter. On the other hand, without the GCH the general problem of the evaluation of κ^λ, where κ and λ are two infinite cardinals, is still open, and we can only define the most basic properties of exponentiation.[31]

The Continuum Hypothesis is probably the most important and famous open problem in set theory. It was first conjectured by Cantor, in the late 19th century, and was the first of David Hilbert's Millennium Problems. The importance of the CH was immediately understood in the mathematical community, and a lot of effort was put in trying to prove it from ZFC.[32] However, it soon became clear that the challenge was impossible: first Gödel proved that $\neg CH$ cannot be proved from ZFC, and then in 1963 Cohen showed how to build a model of ZFC where the CH was true *and* a model of ZFC where the CH was instead *false*,[33] thus proving that the CH was *independent* of the axiom of ZFC. From that moment, set theory underwent a deep transformation: the main effort was in building more of these models (universes), each different from the

31 These can be found in any elementary textbook in set theory, like Hrbacek and Jech (1999).

32 Hilbert's Millennium Problems are a set of open problems that David Hilbert indicated as fundamental for the development of the discipline at the Paris conference of the International Congress of Mathematics. See Hilbert (1902).

33 See Cohen (1964).

others, and in enhancing the canonical set of axioms ZFC so that CH would become provable in it.

So what could be a solution to the continuum problem? What would be needed is either a proof of CH or of its negation. Since no such proof can be obtained in ZFC, Gödel (1947) shifted the focus to adding new axioms that imply the CH, thus proving it, or that imply that the CH is false. This is what is known as *Gödel's Programme*: find an axiom A, that becomes *universally accepted* (by being either self-evident, thus *intrinsically justified*, or with such good consequences that we cannot avoid including it, thus being *extrinsically* justified), such that $\mathsf{ZFC} + A \vdash CH$ or $\mathsf{ZFC} + A \vdash \neg CH$. Since 1947, a large number of candidate extra axioms have been proposed, many of which are mutually incompatible.[34] The upshot, then, is a number of extensions of ZFC, many of which are mutually incompatible.

The hard moral to draw from independence, then, is that universists who believe that there is a single, *determinate* universe of sets (a "single V") may just be wrong: although V clearly reflects the ZFC axioms, multiple V's, so to speak, can be seen as reflecting *alternative extensions* of ZFC and as satisfying set-theoretic statements left undecided by ZFC, such as CH. The independence of CH is then a problem because we don't know whether CH is true in V or not. The plethora of extensions of ZFC that settle CH all paint a very different picture of what the cumulative hierarchy looks like. Which one should we take to be the "true" one? This presents us with a choice: electing one of these theories as "the one" set theory, thus choosing a precise strengthening of V and scraping all the other theories (and universes), or building a multiverse, where all these theories are on the same "plane".

3 A Natural Extension of ZFC: Large Cardinals

The first axiom candidates to extend ZFC and, hopefully, settle the independent question of CH were the Large Cardinal Hypotheses (LCs for short).[35] These axioms state the existence of a particular large cardinal or of an entire class of them.

A large cardinal is a transfinite cardinal number that cannot be reached by any set-theoretic operation. Their size is larger than any other cardinal number that can be defined in ZFC. Moreover, their existence *cannot* be proved from ZFC. Felix Hausdorff (1908) introduced the first large cardinals, the *weakly inaccessible cardinals*. A cardinal is weakly inaccessible iff it is a regular weak limit cardinal, i.e. iff it is equal to its

34 The most notable incompatibilities are the ones between Reinhardt cardinals and Choice (Solovay et al. 1978), inner models and large cardinals (Jensen 1995), and determinacy and Choice (Mycielski et al. 1971).

35 The main and best reference on large cardinals is A. Kanamori (2003).

cofinality[36] and if it's neither a successor cardinal[37] nor zero. After this first seminal definition, a vast number of new large cardinals were introduced. They are usually categorised in several different types accordingly to their properties. The most general classification divides them in "small" large cardinals (i.e. the large cardinals compatible with $V = L$, from the weakly inaccessible up to measurable cardinals) and "large" large cardinals (i.e. the large cardinals incompatible with $V = L$, from measurable upward). Other possible classification includes "combinatorial" large cardinals (between inaccessible and weakly compact), "Ramsey-like" cardinals (between weakly compact and measurable), and the "extender" large cardinals (between measurable and superstrong).

The most intriguing property of LCs is that they form a linearly ordered hierarchy in which the larger large cardinals prove the existence of the smaller ones. For example, a weakly compact cardinal[38] proves the existence of a strongly inaccessible cardinal,[39] and the existence of that weakly compact cardinal is proved by the existence of a Q-indescribable cardinal,[40] and so on (see Figure 2). However, there is a limit of how high we can go in this hierarchy: if we keep defining stronger and stronger large cardinals we end up contradicting the Axiom of Choice. The first large cardinal that does that is the *Reinhardt cardinal*, the first of the *choiceless* cardinals.[41]

This hierarchy of LCs can also be seen as a hierarchy of extensions of ZFC, each of which is the result of adding a LC Hypothesis to ZFC. Given two LC Hypotheses, H_1 and H_2, if $H_1 \Rightarrow H_2$ then $\mathsf{ZFC} + H_1 \Rightarrow Con(\mathsf{ZFC} + H_2)$. Thus LCs can also be used to measure the *consistency strength* of statements and theories. This in turn gives us a hierarchy of theories, linearly ordered by their consistency strength.

Gödel himself, when introducing its famous Programme to settle the CH (Gödel 1947), thought that large cardinals were a legitimate addition to ZFC – one that would eventually help deciding CH. That is, the hope was that there was a large cardinals α such that its corresponding axiom "there exists a large cardinal α", if added to ZFC,

36 The cofinality of a cardinal κ, written $cf(\kappa)$, is the least of the cardinalities of the cofinal subsets of κ (a subset λ of κ is said to be cofinal iff for every $a \in \kappa$, it is possible to find an element $b \in \lambda$ such that $a \leq b$).

37 A successor cardinal is a cardinal that immediately succeeds another cardinal, without any other cardinal between them.

38 A cardinal κ is weakly compact iff it is uncountable and for every function f: $[\kappa]^2 \mapsto \{0, 1\}$ there is a set of cardinality κ on which f is constant.

39 A cardinal κ is strongly inaccessible iff it is uncountable, it is not a sum of fewer than κ cardinals smaller than κ, and $\alpha < \kappa \Rightarrow 2^\alpha < \kappa$. The proof can be found (among other places) in Baumgartner (1977).

40 Q-indescribable cardinals are cardinals that are difficult to axiomatise in some language Q. See Lévy (1971) for details.

41 A cardinal κ is Reinhardt iff it is the first cardinal that doesn't get mapped to itself in a non-trivial, elementary embedding between V and itself: j: $V \mapsto V$. See W. N. Reinhardt (1974) for details. The proof of the inconsistency with AC can be fund in Kunen (1971).

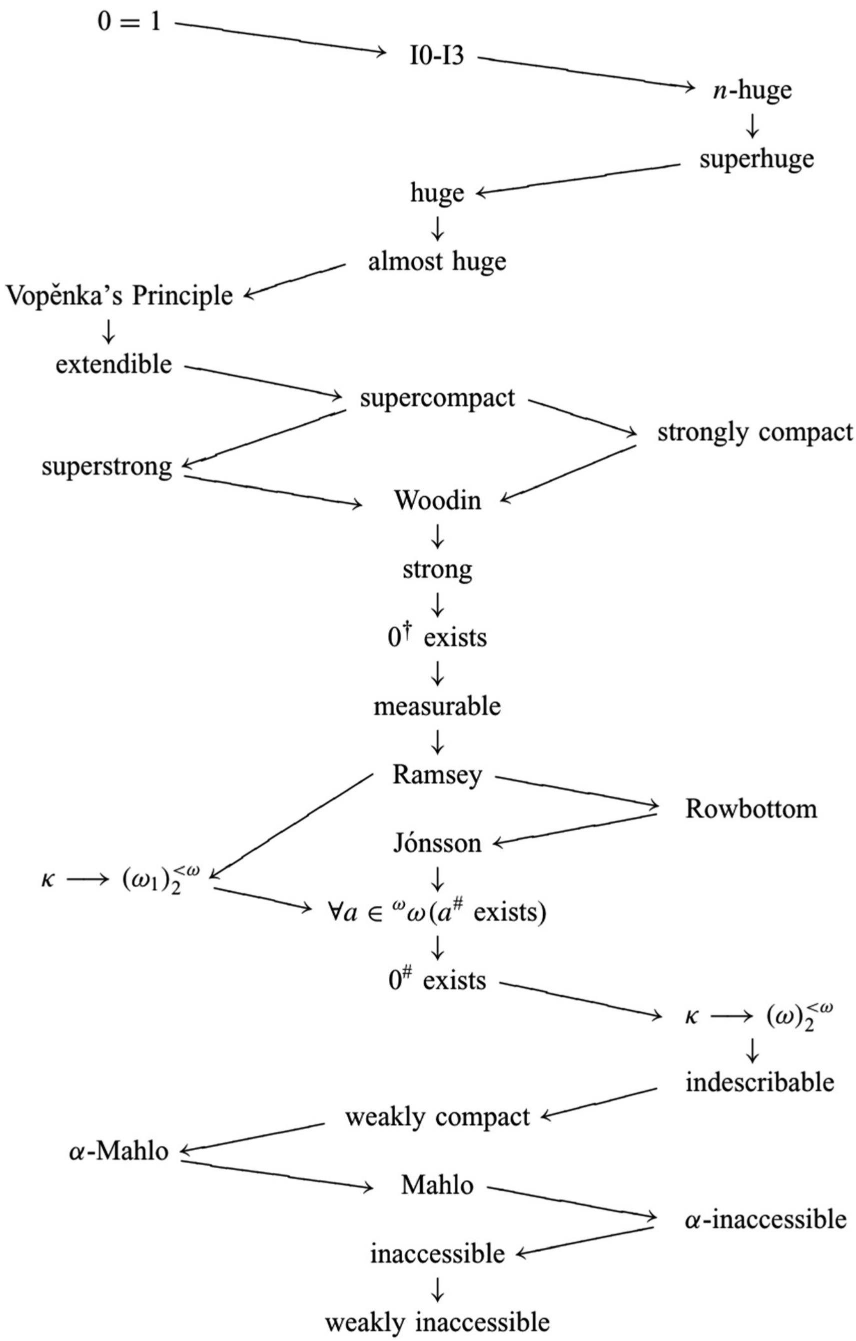

Figure 2: The hierarchy of LCs, from A. Kanamori (2003).

would prove CH (or its negation). Alas, this wasn't possible. As proved by Lévy and Solovay (1967), large cardinals don't settle the CH in neither direction.

Robert Solovay's proof, together with Cohen's forcing method, meant the end of Gödel's Programme. Even though other axioms were proposed in the following years (e.g. Forcing Axioms and the Determinacy axioms[42]), none of them has been generally accepted, unlike LCs, many of which are now seen to be legitimate extensions of ZFC. So we are now left with several possible axioms that can extend ZFC, that are in most cases mutually incompatible. For example, the full Axiom of Determinacy (AD) is incompatible with AC (see Mycielski et al. 1971). So how to conceive of set theory in light of the independence phenomena? There are two main conflicting views, universism and pluralism, which I now briefly outline.

4 Universism

Universism is the thesis that there is only one set-theoretic universe, namely, V. This universe is considered the *intended* model of ZFC and set theory (similarly, $\mathbb{N}$ is considered the intended models of PA and arithmetic), as opposed to all the others models, the *non-canonical* models, such as, for instance, the constructible universe L.

The main philosophical attitude which universists may adopt to substantiate their view may be called *Gödelian Platonism*. According to this, in "constructing" V, we have been guided by a special form of intuition about a *mind-independent* realm of objects. Our mental vision, of course, is not perfect - or else, according to Gödelian platonists themselves, we'd have a completely determinate conception of V, which is something that, even by the Gödelian platonist's lights, we certainly lack. Still, Gödelian platonists typically argue, we can expect our set-theoretic intuition to become sharper in time, so as to provide us with a more exhaustive description of the realm of sets. On such a view, non-canonical models of set theory are not epistemologically on a par with "real V".

The main problem with Gödelian Platonism, however, is that it doesn't seem to have the resources to tell us which of all the possible, mutually incompatible extensions of ZFC is true of the intuitive "concept of set". Indeed, Gödel thought that stronger set-theoretic axioms, based on such a concept, and possibly also *extrinsically* justified, would ultimately fill in the gaps in our knowledge of V (cf. Gödel 1964, pp. 476–77). But, as we saw in the previous sections, after many years of extensive research, no "new" set of axioms has, so far, stood out as the most plausible extension

42 Forcing Axioms gives us conditions on which certain types of forcing are possible or not, while Determinacy Axioms tell us which games are determined or not. A good account of both and their relations is W. H. Woodin (1999).

of ZFC. As a consequence, appealing to a somewhat mysterious faculty of intuition, and to our concept of set, simply doesn't seem good enough: intuition doesn't seem very reliable, and – it may be argued – we have every reason to think that our concept of set may rather be indeterminate.[43]

5 The Set-Theoretic Multiverse

As already mentioned, in recent years a new view regarding the problem of extending ZFC has started to gain prominence: *pluralism*. According to this view, there is no such thing as *the* "one true set theory" and *the* "one true set-theoretic universe". Instead, set theory is the study of all the possible, and different, axiomatizations of set theory, and of their corresponding models. From a philosophical perspective, this resembles the "non-euclidean" turn in geometry at the start of the XIX Century. At that time, Euclidean Geometry ceased to be considered the one true geometry, and different geometries started being developed and studied. All these different geometries were then considered equally legitimate. Although some of them may be preferable, this can only be for pragmatical reasons, not because they are "more correct" than the others. As Henri Poincaré put it:

> There is no such thing as a true geometry, only a more convenient one. [(Poincaré 2012, p. 85)]

The set-theoretic pluralists propose to make exactly the same turn in set theory, as argued by J. Hamkins (2012).

However, even assuming that the pluralist's philosophical goals are worth pursuing, that doesn't mean that there is a corresponding mathematical framework that instantiates those philosophical ideas. Indeed, while other branch of mathematics enjoy such frameworks (e.g. algebra for all the kinds of geometries, universal algebra for algebras, universal logic for logic, etc.), the search for a corresponding framework for the different set theories is still ongoing: several different set-theoretic multiverses have been proposed.

5.1 The Philosophical View: Pluralism

Pluralism in set theory has long been advocated, and defended, by Hamkins (see especially J. Hamkins 2012). According to such a conception, there is no such thing as a

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43 For an attempt to offer an epistemologically more adequate platonist account of set theory, see e.g. Tieszen (2012). Platonism faces of course other difficulties; see e.g. Benacerraf (1965, 1973) and Field (1989).

single, true theory of sets instantiated by a single canonical model. Such a situation is made impossible by the independence phenomenon: any theory of sets can be extended in several, mutually incompatible ways, each one with its own collection of models. Likewise, its supposed canonical model can also be extended in incompatible directions. As Hamkins writes:

> [T]he most prominent phenomenon in set theory has been the discovery of a shocking diversity of set-theoretic possibilities. Our most powerful set-theoretic tools, such as forcing, ultrapowers, and canonical inner models, are most naturally and directly understood as methods of constructing alternative set-theoretic universes. A large part of set theory over the past half-century has been about constructing as many different models of set theory as possible. [(J. Hamkins 2012, p. 418)]

Faced with such an embarrassment of riches, with countless universes and theories to choose from, it is impossible to pin down the one true theory of sets, with its associated canonical model; we can only settle for a *plurality* of theories, models and truths, without trying to determine the "truest" between them. According to Hamkins, *all* set theories and their models are part of the multiverse. This gives us the most variety of possible universes and theories. And, crucially, all such possible universes have equal status. That is, no universe is metaphysically privileged. Moreover, there is no criterion to select between universes, so every imaginable universe is part of this kind of multiverse. In Hamkins' view, the main job of set theory is to deal with different kinds of constructions, which can verify, or falsify, the same set-theoretic claims (such as CH). Consequently, there is no reason to ban a particular universe: they are all perfectly legitimate model theoretic constructions. Such a view is clearly reminiscent of Poincaré's reflections on geometry, quoted above. And, indeed, Hamkins explicitly states that his goal is to transform set theory in a "geometry-like" theory:

> There is a very strong analogy between the multiverse view in set theory and the most commonly held views about the nature of geometry. [...] At first, these alternative geometries were considered as curiosities, useful perhaps for independence results, for with them one can prove that the parallel postulate is not provable from the other axioms. In time, however, geometers gained experience in the alternative geometries, developing intuitions about what it is like to live in them, and gradually they accepted the alternatives as geometrically meaningful. Today, geometers have a deep understanding of the alternative geometries, which ar regarded as fully real and geometrical. The situation with set theory is the same. [...] Like the initial reactions to non- Euclidean geometry, the universe view regards these alternative universes as not fully real, while granting their usefulness for proving independence results. Meanwhile, set theorists continued, like the geometers a century ago, to gain experience living in the alternative set-theoretic worlds, and the multiverse view now makes the same step in set theory that geometers ultimately made long ago, namely, to accept the alternative worlds as fully real. [(p.425 J. Hamkins 2012, p. 426)]

Such a view equates the independent statements of set theory to Euclid's Fifth Postulate, or to the Commutative Axiom from group theory: they are not essential to the nature of what geometry or group theory is, but only one of the properties that *a* geometry or *a* group can have. In set-theoretic terms, this means that it doesn't matter what the "real" truth value of the CH is, or what axioms we can add to ZFC to settle it. We have already settled it by knowing in which universes CH holds and in which ones it doesn't. The satisfaction of CH (and the size of the continuum) becomes just one of the features of a universe: for example, we might have universes in which the continuum has size $\aleph_1$ (because CH here is satisfied), or $\aleph_2$ (because, in such a universe, PFA holds instead), and so on.

Philosophically speaking, this view is quite appealing and intuitive. It solves the main problem of independence in an elegant way, and it also opens up several new avenues of research. For example, the recent contributions to the modal theory of forcing or set-theoretic geology were both spearheaded by the pluralist idea.[44] However, the question arises how to mathematically specify a multiverse framework – a unified, pluralist theory of sets. Indeed, pluralism in the foundations of mathematics can be instantiated in a number of radically different ways. Each one of these characterisations, i.e. each set-theoretic multiverse, is unique and comes with its own particular variety of pluralism.

5.2 The Mathematical Characterisation

From a philosophical point of view, we can classify the various types of multiverses by their commitment to the ontological existence of the universes. More precisely, we can recognise two main views:

- a *realist* multiverse, committed to the full existence of the universes that form it (e.g. Mark Balaguer's full-blooded Platonism[45]);
- an *anti-realist* multiverse, that isn't committed to the platonic existence of the universes (see e.g. Shelah 2003).

Instead of focussing on the ontological existence of the universes, the mathematical classification is based on which universes are part of the multiverse. Thus, we can have:

- the *broad* (or radical) multiverse, formed by all possible universes without any restriction (see e.g. J. Hamkins 2012);
- a *generic* multiverse, where new universes are built using forcing; we can then have several variants of the generic multiverse, like Hugh Woodin's set

44 See especially Fuchs et al. (2015) and J. Hamkins and Øystein Linnebo (2022).

45 Cfr. Balaguer (1995, 1998).

generic multiverse that uses the very strong Ω-logic as its background logic (W. H. Woodin 2011b), or Steel's set generic multiverse, that supposes the existence of a *core* (Steel 2014);[46]
- a *parallel* multiverse, where all the universes instantiate a different powerset operation (Väänänen 2014);
- the *Hyperuniverse* (HP), that is, the collection of all the countable transitive models of ZFC (see Arrigoni and S.-D. Friedman 2013).

At a general, informal level, a set-theoretic multiverse can be described as a collection of models of set theories. Given some consistent set theories $T, T', \ldots$, the models of $T, T', \ldots$, are part of the multiverse. Each model (intended in the usual way of a domain together with an interpretation function) represents a set-theoretic universe. Consequently, each consistent theory determines at least one universe in the multiverse. In the multiverse one can find models of more than one theory. For example, one multiverse can be defined as the collection of all countable transitive models of ZFC, thus in that multiverse one can find universes from the models of, say, ZFC, ZFC + $\neg CH$, ZFC + CH, ZFC + $V = L$, etc. These models jointly capture all there is to know about sets. That is, each model is a legitimate universe of set theory and, as a consequence, there is no Single Universe. According to proponents of the multiverse, this lack of unity cannot be repaired in any way and set theory is precisely the study of all these alternative universes, in which the properties of sets can vary greatly from one to another.

Such a conception of the multiverse is *semantic*. There might be several, mutually incompatible *models* of ZFC. The set-theoretic multiverse is a collection of width extensions $V[G]$ of V such that $V[G]$ satisfies a statement φ, that's indeterminate in V (CH is the prime example of such a φ). In most cases, these models are produced by the application of forcing. Thus, there is a clear connection between pluralism and potentialism: in order to be able to take the collection of all extensions of V, we need to assume that V is indeed extendible. And since these extensions are usually produced through forcing, and with forcing we add subsets to V, width potentialism is a needed assumption. It is also possible to assume height potentialism, and include in the multiverse also height extensions of V (albeit multiverses that do this are much rarer, and multiverses that include *only* height extensions haven't been developed in the literature). Radical potentialism (that allows extensions both in height and width) is obviously also present in the several possible characterisations of the multiverse. Nevertheless, width potentialism remains the most common view. This is not to say

46 Other possible generic universes are obtained changing the type of forcing we are using to build them, so we can have a generic multiverse for each different forcing approach (class-generic, hyperclass, Robinson's infinite forcing, etc.).

that an actualist pluralism is in principle impossible. It would be perfectly fine to defend the view that there are several set-theoretic universes that cannot be extended neither in height nor in width. According to this view, one cannot use forcing to produce new extensions of the universe, thus one would need a procedure to generate these different universes. This hasn't been really explored in the literature, but a possible way to approach such an actualist multiverse would be to define each universe as generated by different powerset operations (in a way similar to Väänänen 2014). On the other hand, actualism is much better connected to universism: if the universe is unique and *the* intended model of ZFC, then assuming that it is also un-extendible is a very natural step.

Truth in the set-theoretic multiverse is usually conceived in a *supervaluationist* way. First, truth is defined *relatively* to a universe: that is, a sentence φ will be true-in-U or false-in-U, where U is a universe of the multiverse. Then, define a notion of truth across the whole multiverse: a statement φ is True iff it is true-in-U for all U in the multiverse (and False if it's false-in-U for all U in the multiverse). A sentence φ is *determined* if it has a truth value across the whole multiverse: so, if and only if for all U in the multiverse, φ is either true-in-U or false-in-U. If a statement is True (or False) then it is also determined in the multiverse (but not the other way around). For example, if we take the multiverse to be the collection of all models that settle CH, we have that CH is determined in the multiverse, since it will have a truth value in all universes of the multiverse. However, it will be neither True or False *in the multiverse*, since there will be both universes in which CH is true-in-U, and universes in which it's false-in-U. On the other hand, if we take the multiverse to be the collection of all countable transitive models of ZFC, then CH will be neither True (or False), nor determined, since there will also be universes in the multiverse in which CH is not settled.

6 Outline of the Contributions

A little more than a decade ago, in 2012, Joel Hamkins published his seminal paper on the set-theoretic multiverse (J. Hamkins 2012), introducing the concept of a mathematical characterisation of the most radical pluralist ideas about the models of set theory. While mathematical pluralism wasn't a new idea at the time (it can already be found in Carnap et al (1987)'s *tolerance principle* for logic, and in Balaguer (1995)'s *full-bloaded platonism*), Hamkins' paper was the first that tried to make the idea mathematically precise. Following his publication, several other papers developed and investigated the idea, both from a philosophical and mathematically perspective. New set-theoretic multiverses were being defined (see W. H. Woodin 2011a; S.-D. Friedman 2012; Steel 2014; Väänänen 2014), and the old ones have been analysed in detail (Bagaria and Ternullo 2023; Koellner 2013; Maddy and Meadows 2020; Meadows 2021). A new, and fruitful, research field was born. Now more than 10 years have passed, there is quite the corpus of papers on the matter, and several connection between the set-theoretic multiverses and other, more traditional, branches of the philosophy of set theory (e.g. potentialism, the nature of forcing, and the problem of axiom selection, to name just a few) have been drawn and investigated. Times are thus mature for taking stock, and assess the debate in both its mathematical and philosophical results.

This special issue is based upon the Workshop "Reflecting on 10 Years in the Set Theoretic Multiverse", organised inside the 2022 SoPhiA Conference in Salzburg. The idea of the workshop, and consequently of this special issue, was to put together a number of novel contributions on the set-theoretic multiverse, from both the mathematical and philosophical perspective. The workshop was a success: new mathematical results were presented, and the philosophical assumptions behind the very idea of a set-theoretic multiverse clarified. The aim of this special issue is to continue that discussion, and present new mathematical results, and new philosophical ideas, connected with the set-theoretic multiverse.

The contributions included in this special issue range from purely mathematical ones that presents novel results (Hamkins 2025), to more philosophical ones that connect the set-theoretic multiverse to mathematical practice (Kant 2025) and the problem of measurament (S. Friedman and Edwards 2025). Finally, Serikaya, Perez-Escobar, and Rittberg's paper (Pérez-Escobar et al. 2025) investigates the problem of the set-theoretic multiverse through the lenses of the late Wittgenstein and petrification. In what follows, I will introduce each paper in more detail, knowing that I cannot make full justice to them and their arguments.

The first paper of this collection is Joel Hamkins' paper, titled *Every Countable Model of Arithmetic or Set Theory has a Pointwise-Definable End Extension*. The main topic of the paper is some novel theorems, one for *PA* and the other for *ZF* set theory, regarding pointwise definable models and their extensions. Pointwise definable models are, roughly speaking, models in which each element can be individually defined without parameters. For example, the set of natural numbers $\mathbb{N}$ is a pointwise definable model of *PA*, since every element of $\mathbb{N}$ can be individually defined starting from 0 and using the successor function. Intuitively, it seems that the set of real numbers, if taken as a model of a theory (e.g. of *ZF* set theory), shouldn't be pointwise definable, since there are uncountable real numbers but only a countable amount of possible definitions. Hamkins' results challenge this intuition: there are pointwise definable models of *ZF*, and they can even be extended to bigger, still pointwise definable models. To prove these results, Hamkins uses the Universal Argument, a Turing Machine that enumerates sequences of numbers first developed by W Hugh Woodin and Heller (2011). The same method, that works for *PA*, can be extended to the case of *ZF*, with just some adjustments. These results gives us a novel perspective to the toy model of the multiverse as developed by Gitman and J. Hamkins (2011). In particular, with these results it becomes possible investigating smaller, countable models as *simulacra* of the big, uncountable models that are full blown universes inside the set-theoretic multiverse. Moreover, the results about the expandability of pointwise definable models also lean in the potentialist conception of set theory, building another connection between the field of set-theoretic pluralism and that of set-theoretic potentialism.

Kant's paper *The Multiverse View and Set-Theoretic Practice* follows Hamkins' paper. Her goal is to investigate Hamkins' proposal of the set-theoretic multiverse, as opposed to the single universe view, in the context of set-theoretic practice. To do so, she uses *quantitative* methods, following a number of interviews with set theorists (28, to be precise). Such an approach is quite innovative in the context of the foundations of mathematics, and only recently is starting to gather some favour in the philosophy of mathematical practice (a similar approach can be found in Tanswell and Inglis (2023), that uses computational linguistics methods). The first result that Kant can gather from the interview is that set-theoretic practice, maybe unsurprisingly with hindsight, is more fractured than usually claimed in arguments supporting the set-theoretic multiverse (that usually appeal to it in a very general and universal way). In particular, there are pluralist-aligned branches of set theory that obviously align with the arguments in favour of the set-theoretic multiverse (e.g. the exploration of forcing), but also absolutist branches of set theory, whose practitioners are more in tune with the universe view (e.g. descriptive set theory). However, Kant shows that some generalisations can still be made across both the universists and the pluralists. First, both groups believe that Hamkins' proposal is too radical, and that his set-theoretic multiverse fails because it's too heterogeneous (a similar objection can be found in Koellner 2013). But, on the other hand, they agree with Hamkins' argument that no axiom that settles *CH* will ever be widely accepted as the "right" extension of *ZFC*, even if for different reasons (some of them because they believe that no axiom can extend *ZFC*). All in all, this paper enlightens how Hamkins' proposal has been received in the set theorists' community. Such a perspective is rare and very valuable, since most of the debate surrounding the set-theoretic multiverse has been mainly philosophical.

The next paper is from Friedman and Miller Edwards, and it focusses on *Measurement in Set Theory*. The starting point of the paper is that in scientific practice and epistemology facts verified by some kind of measurement are considered more reliable than facts derived from "unmeasurable" theories. However, scientific advancements during the 20th century (especially the theory of relativity) showed that any measurement is only relative to a frame of reference. The main results that Friedman and Miller Edwards argue for is that some fundamental functions in set theory (mainly forcing and other methods aimed at the definition and construction of models) function in a similar way to measurement in science, and just like measurement in science they are relative to soma frame of reference. Such an argument agrees with some of J. Hamkins (2012)'s remarks on the *perspectivism* of the set-theoretic multiverse: as seen from a particular universe, all other universes seem to be countable, ill-founded sets. Friedman and Miller Edwards apply categories usually focussed on the investigation of scientific models to the debate between the multiverse view and the single universe view. They notice that there are several similarities between how we approach the study of scientific models and the plethora of set-theoretic models. This is quite far from the usual appeal to the *a priori* in the logical, and mathematical, context, and can be without a doubt be a very constructive way to approach a debate usually argued on very different grounds.

Finally, Serikaya, Pérez-Escobar, and Rittberg's paper *Petrification in Contemporary Set Theory: The Multiverse and the Later Wittgenstein* concludes this special issue. The main goal of this paper is to connect the set-theoretic multiverse with issues in the philosophy of mathematical practice and hinge epistemology. In particular, they use the late Wittgensteinian notion of *petrification* as the main way to explain some phenomena in advanced mathematics (in this context, in reality it could be applied wildly from epistemology to the philosophy of science). According to the late Wittgenstein (Wittgenstein 1956), the more we experience some phenomenon, the more it "petrifies" from purely empirical to a normative *hinge* (e.g. we keep noticing that adding 2 thing to 2 other things we get 4 things, and this then petrifies in the normative judgement that $2 + 2 = 4$). The authors apply this notion to current disagreement in the absoluteness of *CH*. Their argument shows that since set theorists have experiences models in which *CH* is true and models in which it is false since the development of forcing by Cohen, now the non-absoluteness of *CH* has "petrified" in set-theoretical practice, and cannot revert back to be considered absolute. This perfectly aligns with one of the main motivations behind the development of the set-theoretic multiverse in J. Hamkins (2012). According to J. D. Hamkins (2015), a dream solution to *CH*, i.e. a solution that assumes the absoluteness of *CH*, is no longer possible, since it would disregard the set theorists experience in the whole range of models with *CH* or without it. Notice however that the authors also argue that not all hinges are universal, and this clashes with some of Hamkins' most radical views). This paper does a great job in clarifying the deep philosophical roots of Hamkins' arguments and of the motivations behind the development of the set-theoretic multiverse. Moreover, it connects the debate around the multiverse with the philosophy of mathematical practice and hinge epistemology, two research fields that are quite novel (see Mancosu (2008) and Coliva (2016)) and booming at the moment.

In conclusion, this special issue gathers new results and new approaches to one of the newest, and most exciting, research field in the foundations of mathematics. What I believe stands out is not only the quality of the contributions (but this is not my judgement to make), but also the fact that all contributions connected the set-theoretic multiverse to other well-researched areas (the philosophy of mathematical practice, philosophy of science, hinge epistemology). It is my hope that such a decision from the authors will stands out in the literature, and will raise new and interesting questions in the debate.

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