



## Book Reviews

STEFANIA CENTRONE, DEBORAH KANT, DENIZ SERIKAYA, **Reflections on the Foundations of Mathematics. Univalent Foundations, Set Theory and General Thoughts**, vol. 407 of *Synthese Library*, Springer, 2019, pp. 494+xxviii; ISBN: 978-3-030-15654-1 (Hardcover) 149.79€, ISBN: 978-3-030-15655-8 (eBook).

The foundations of mathematics is one of the central research areas in the philosophy of mathematics since the late 19th century. The volume *Reflections on the Foundations of Mathematics*, edited by S. Centrone, D. Kant, and D. Serikaya (Springer, 2019), aims at collecting papers contributing to the main topics of current debates in the foundations of mathematics. The volume originates from the conference *Foundations of Mathematics: Univalent Foundations and Set Theory (FOMUS)*, which was held at Bielefeld University in July 2016. The contributions are split between more mathematical topics (e.g. the development of new set-theoretic multiverses, the investigation of category theory and homotopy type theory, the formalisation of mathematics in languages understandable by automatic proof assistants, and so on) and more philosophical ones (e.g. the importance of mathematical practice, how to evaluate a good foundations of mathematics, the question of pluralism). Obviously, the boundaries are not sharp, and mathematically-minded philosophers end up working with philosophically-minded mathematicians in practically all of these topics.

After an informative introduction, that provides some historical and mathematical background on the topics of the collection, Part 1, *Current challenges for the set-theoretic foundations*, is devoted to the current debates in set-theoretic foundations. with particular focus on actual mathematical practice (Džamonja and Kant [12]), the choice of new axioms (Fontanella [14]), the debate on the set-theoretic multiverse (Ternullo[27]), and the interplay between independence and reflection (Welch [29]). Part 2, *What are Homotopy Type Theory and the Univalent Foundations?*, focuses on Homotopy Type Theory and Univalent Foundations, an alternative approach (that is becoming quite popular) to set-theoretic foundations. In this part, the papers range from introductory papers on the main notions of HoTT and Univalent Foundations (Altenkirch [2] and Ahrens and North [1]), to problems regarding the choice of models of HoTT (Buchholtz [7] and Rodin [24]), and the formalisation of HoTT in the Coq programming language

(Bordg [5]). Part 3, *Comparing set theory, category theory, and type theory*, compares the foundational views of the three main candidates to be a foundations of mathematics: set theory with category theory (Barton and Friedman [4]), set theory with type theory (Klev [18]), and some general comparisons (Džamonja [11] and Maddy [21]). Part 4, *Philosophical thoughts on the foundations of mathematics*, is more philosophical, and focusses on the debates of automatic proofs and proof checking (Carl [8]), mathematical pluralism (Friend [15] and Priest [23]), and the need of a foundations of mathematics from the perspective of the working mathematician (Wagner [28]). Finally, Part 5 introduces the reader to themes from the philosophy of mathematical practice, with particular attention to its bearings on the foundational debate. Here the topics vary from another comparison of set theory and category theory from the perspective of mathematical practice (Bowler [6]), to questions regarding proof representation (Fisseni et al. [13]), the development of automatic proof assistants (Paulson [22]), and finally a novel approach to the question of the foundations of mathematics (Sambin [25]).

As it can be seen from this brief summary, this volume is incredibly rich and varied. It contains a wealth of exciting and thought-provoking contributions that I could not plausibly do justice to all of them in such a limited space. I limit myself to making a few general comments, and point to some very interesting results that are presented in this volume.

This volume is invaluable for the anyone involved in foundational research, since it can be used both to explore new ideas and to have a good bird's view of the whole research field. Even the motivated graduate student, or scholar from adjacent fields can find some interest in this collection, since it can use it as a first overview of the most recent debates in the field. However, a word of advice is needed for someone without too much background in any of the topics presented here: these are not introductory level contributions, so a good mathematical background is certainly required. This said, the introduction does its job and introduces some preliminary notions, thereby helping any newcomers getting up to speed. This might be especially true for contributions on Homotopy Type Theory and category theory (Part 2 and 3).

The main theme that serve as the silver lining of the whole volume is the attention to mathematical practice that we need to devote even when investigating matters in the foundations of mathematics. This becomes clear already with the first paper of the collection, Džamonja and Kant [12]. In this paper, the second author interviews the first one on questions about the independence phenomenon in set theory, the history of forcing, and how to choose our set-theoretic axioms. These reflections are all from the

perspective of a working set theorist, not a philosopher of mathematics, and this gives rise to some interesting, and novel, statements. For example, towards the end of the paper, 3 hypotheses are put down, to try to summarise the findings of the interview:

- H1 Most set theorists were surprised by the introduction of forcing;
- H2 Most set theorists think that forcing is a natural part of contemporary set theory;
- H3 Most set theorists prefer an answer to a problem with the help of a new axiom of *lowest possible consistency strength*. And for most set theorists, a difference in consistency strength weighs much more than other differences.

The last one is quite surprising for the philosopher: this is a completely different approach to the most common method in philosophical debates around the axioms, i.e. trying to justify the new axiom intrinsically or extrinsically (idea that started with Gödel himself). This has an interesting consequence: when comparing the Forcing Axioms<sup>1</sup> and the Large Cardinals Axiom,<sup>2</sup> it seems that the working mathematician doesn't care about the differences between their "natures" or their consequences to other problems, but considers only which one she can use to prove a given theorem, and which one has the lower consistency strength. Again, this is surprising for the philosopher. A philosopher of mathematics is used to think in "foundational" terms: one needs to develop a foundations of mathematics strong enough to settle independent questions, while avoiding the paradoxes. Since the most prominent of these questions is the Continuum Hypothesis (CH), each new addition to the axioms of our foundations of mathematics comes with the question "How does this new axiom settle CH"? Here the philosopher would be wary of adding Forcing Axioms, if she believes that CH should be true. This is the main reason why Large Cardinals Axioms are usually preferred (an example of such preference can be seen in Steel [26] and Maddy, Makowsky,

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<sup>1</sup>Forcing Axioms specify conditions on which certain types of forcing are possible or not. For example, the Proper Forcing Axiom states that if a proper partially ordered set  $P$  is *proper* (i.e. if for all regular uncountable cardinals  $\lambda$ , forcing with  $P$  preserves stationary subsets of  $[\lambda]^\omega$ ) and  $D_\alpha$  is a dense subset of  $P$  for each  $\alpha < \omega_1$ , then there is a filter  $G \subseteq P$  such that  $D_\alpha \cap G$  is non-empty for all  $\alpha < \omega_1$ . Crucially, it implies the negation of CH, with  $2^{\aleph_0} = \aleph_2$ , see Asperó, Larson, and Moore [3].

<sup>2</sup>Large Cardinals Axioms simply state the existence of some large cardinal. For example, they might state that measurable cardinals exist, or that there is a proper class of Woodin's cardinal, etc.. None of them settle CH.

and Ravve [20]). However, from this paper we can conclude that no such preference can be actually found in mathematical practice.

The attention to mathematical practice that is professed in Džamonja and Kant [12] can be seen influencing most of the other papers in the volume. For example, Fontanella [14] discusses current arguments in favour and against prominent axiom candidates (the axiom of constructability, large cardinals axioms, forcing axioms, ultimate- $L$ , and determinacy axioms). However, after having briefly described the classical position on intrinsic and extrinsic justification, she completely disregards it, preferring to take into consideration how these axioms could be used for mathematical practice. Another good example is Maddy [21], in which set theory, category theory, and Homotopy Type Theory are compared as possible candidate foundations of mathematics. This comparison is carried out by looking at the actual use of a foundation of mathematics in mathematical practice, and not at some metaphysical ideas (in line with the well-known Maddy's naturalism from Maddy [19]).

Another important focus of the volume is the recent debate around the set-theoretic multiverse and pluralism. The approach is mainly philosophical, with (again) particular attention to mathematical practice. There are several papers on this topic scattered through all parts. The fact that this is a timely and hot debate is witnessed by the addition to the volume of a reply to one of the contributions (Ternullo [27]). Both contributors offer very good and compelling arguments, and a very good job in pointing the main turns in the dialectic. Here we can maybe find a missed chance for the volume: adding more replies to other contributions instead of just one. More interaction between the contributions could have been an even better ad for the research field, especially in some cases (for example in Part 3, the more "dialectic" of this collection). However, the editors surely had to make some difficult choice considering that the volume is already quite long (over 500 pages), and they clearly did a terrific job in the overall choice of which contributions to include, so this point cannot really be considered a flaw.

I must recommend especially the part on the philosophy of mathematical practice (Part 5). The topic is so new and of recent interest that collecting contributions on it, generally aimed at the researcher on the foundations of mathematics, was really needed. I found particularly interesting the focus on automated proving methods, for example the Isabelle/HOL programming language (a higher order logic version of Isabelle, derived from the HOL Light language) from Paulson [22], or the Coq programming language used for the UniMath Library from Bordg [5]. In both cases, the main novelty is trying to apply automated proof systems to the area of the foundations of

mathematics. Paulson [22] manages to show that simple type theory is sufficient to do significant mathematics by using it to translate its core principles and proofs in the Isabelle/HOL programming language. The main point of his result is that it is possible to use simple type theory and higher order languages to formalise even real analysis *without* having to resort to too many set constraints (needed to simulate dependent types, that are not allowed in simple type theory). Harrison [17] rejects this method because it collapses back to a sort of set theory, but Paulson manages to show that, by building on the idea of *polymorphism*,<sup>3</sup> and *axiomatic type classes*,<sup>4</sup> the problem can be avoided. To show that this is indeed possible, he translates the proof of a theorem on stereographic projections.<sup>5</sup> Paulson's result is even better if we consider that the method he uses to implement the translation in Isabelle/HOL is not too formalism specific, and can be easily translated in different formalisms. He highlights this by noting Gödel [16]'s remarks that reducing all of mathematics to a single formalism is impossible. This is by itself a very interesting novelty, that will certainly bring a paradigm shift in this research field.

As a final remark, I noticed the lack of a final, general bibliography, and of a general index. This makes using the volume as reference material and for brief consultations a little difficult, especially considering the vast variety of topics discussed (such a problem is however less pronounced when using electronic versions).

In conclusion, this collection is as timely as it can be, and the editors clearly did an amazing job in putting it together. It is a fine addition to any departmental library, and any scholar interested in these topics will find new interesting ideas in all the contributions of this volume.

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<sup>3</sup>Polymorphism is essentially *typed ambiguity* that is, the possibility of applying type variables to more than one type. In this way we can have *type schema*: a definition or theorem involving one type variable can stand for all possible instances where types are substituted for type variables. See de Bruijn [10].

<sup>4</sup>Building on the notion of polymorphism and type schema, we can then define a type class as all those types that satisfy a number of general type schema. See Wenzel [30].

<sup>5</sup>Such theorems involve the projection of a plane on a sphere or other curved surface. For Paulson's argument the detail of the theorem don't matter, it's just an example of the power of his formalisation. However, a good exposition of this area can be found in Casselman [9].

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