# A naturalist justification of the Generic Multiverse with a core

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**Abstract**

In this paper, I argue that a naturalist approach in philosophy of mathematics justifies a pluralist conception of set theory. For the pluralist, there is not a Single Universe, but there is rather a Multiverse, composed by a plurality of universes generated by various set theories. In order to justify a pluralistic approach to sets, I apply the two naturalistic principles developed by Penelope Maddy (cfr. Maddy (1997)), UNIFY and MAXIMIZE, and analyze through them the potential of the set theoretic multiverse to be the best framework for mathematical practice. According to UNIFY, an adequate set theory should be foundational, in the sense that it should allow one to represent all the currently accepted mathematical theories. As for MAXIMIZE, this states that any adequate set theory should be as powerful as possible, allowing one to prove as many results and isomorphisms as possible. In a recent paper, Maddy (2017) has argued that this two principle justify $ZFC$ as the best framework for mathematical practice. I argue that, *pace* Maddy, these two principles justify a multiverse conception of set theory, more precisely, the generic multiverse with a core ($GM_H$).

## 1 The Multiverse

The concept of "multiverse" was born following the discovery of the phenomenon of independence in set theory: sentences in the language of set theory, such as the Continuum Hypothesis ($CH$), that turned to be independent from the axioms of $ZFC$. In order to prove these independence results, set theorists make use of *models* (universes) different from the canonical one: the collection of all these models (universes) constitutes the multiverse.

The multiverse then consists of all the models that satisfy the axioms of certain set theory: a very liberal notion of the multiverse would admit every possible universe, for example both $ZFC$ and $ZF + \neg C$, while others would define some criterion, e.g. all the universe without Choice ($ZF + \neg C, ZF + AD$ and so on). In addition, these models contain all the relevant information on sets. Each of these models is a legitimate universe of set theory; there is no Single Universe. According to proponents of the multiverse, this lack of unity cannot be repaired in any way and set theory is precisely the study of these alternative universes, in which the properties of sets can vary greatly from one to another.

From a philosophical point of view, we can classify the various types of multiverse by their commitment to the ontological existence of the universes. More precisely, we can recognize two main positions:

- a *realist* multiverse, committed to the full existence of the universes that form it (e.g. Balaguer's fullblooded Platonism, cfr. Balaguer (1995) and Balaguer (1998));

- an *anti-realist* multiverse, that does not commit to the platonic existence of the universes (this is, for instance, the position defended by Shelah in Shelah (2002) and Shelah (2003)).

Instead of focusing on the existence of the universes, the mathematical classification is based on how we build a hierarchy of all these universes. So we can have:

- a *broad* multiverse, where there is no hierarchy at all, and all the universes have the same status among the others (for instance, Hamkins' multiverse is a broad multiverse, cfr. Hamkins (2012));

- a *generic* multiverse, where we differentiate between universes using a strong logic (an idea owed to Woodin, in Woodin (2011), from now on $GM_\Omega$) or supposing the existence of a *core* (an idea owed to Steel, cfr. Steel (2014), from now on $GM_H$);

- a *vertical* (or *horizontal*) multiverse, where all the universes are like on a ladder, bigger and bigger (or on the same level, but we add more and more universes of the same size);

- an *hyperverse*, that is, a multiverse of multiverse (Arrigoni and Friedman (2013)).

One can of course apply the philosophical classification to the mathematical one, which yields a realist and anti-realist version of all these multiverses.

Not all these multiverses are equally appealing. Some are philosophically weak; others are mathematically trivial. All things considered, a consensus has emerged to the effect that the $GM_H$ is to be regarded as the most promising option. It is very intuitive, and hence philosophically well motivated. And it is mathematically productive: it validates all the theorems of $ZFC$, and beyond.

# 2 The naturalistic approach in philosophy of mathematics

Naturalism in philosophy of mathematics is a methodological approach that recommends using mathematical rules and arguments to justify mathematical objects (usually axioms) and statements about mathematics. In most cases, this means that a new axiom (or theory) should be tested on the basis of its actual or potential benefits for mathematical practice. Every single argument about mathematics should have a specific mathematical goal. To put this definition in a more precise fashion, Maddy (1997) and Maddy (2011) proposed the following principles as a method to test the strength of new axioms (and, subsequently, theories):

**Principle 1 (UNIFY)** *The ultimate goal should be a theory where every structure and every mathematical object can be modelled.*

This essentially says that an adequate set theory should be foundational, in the sense that it should allow one to model most (better: all) of current mathematical practice.

**Principle 2 (MAXIMIZE)** *The new axioms should be as powerful as possible, providing all possible results and isomorphisms.*

A larger number of possible isomorphisms means that our framework has a strong capacity to describe mathematical objects and their relationships.

Maddy (2017) further refine UNIFY, and suggests that a theory is foundational if and only if it provides the following:

- Meta-mathematical Corral;
- Elucidation;
- Shared Standard;

- Risk Assessment.

The first thing that comes to mind when talking about a "foundational" theory is the possibility to embed all of mathematics in a single theory, that is, we should be able to prove, in our foundational theory, something general about all mathematics (Meta-mathematical Corral).

Another important role of the foundational theory is to replace a vague notion with a more precise one. For example, consider the notion of continuity which, before Dedekind's work, was somewhat ambiguous. It was precise enough to generate the calculus, but not enough to become a tool in the proof of fundamental theorems. Only after it was defined in set-theoretic terms it became certain and useful for any purpose.

A foundational theory $T$ should provide us with a Shared Standard to decide if a proof is actually part of mathematics or not. Put differently, $T$ should be the "judge" of what counts as a proof in mathematics and what not: a formal derivation in the foundational theory should be regarded as the "standard" proof in mathematics.

Finally, we should be able to use our foundational theory to assess any new mathematical object that is considered dangerous and suspicious, and to determine how risky it is to use it.

Maddy's MAXIMIZE can be rephrased in such a way as to relate it to Shared Standard (cfr. Maddy (2017)): the foundational theory should be a Generous Arena in which we can hope to find, study and analyze every mathematical object. According to the revised principle, all the possible mathematical structures can co-exist in the foundational theory, and their interrelations can be studied in such a theory. Two observations are in order. First, MAXIMIZE* doesn't go beyond Maddy's original principles, it is just a way of articulating them differently. Second, it is worth noting that the reference to isomorphisms contained in MAXIMIZE is still present in MAXIMIZE*: if all the various structures from mathematics can co-exist in a single framework, the total number of isomorphisms grows.

# 3 A justification of the Generic Multiverse with a core

Maddy (2017) has recently argued that, from a naturalist point of view, the only possible choice as a foundation for mathematics is the Single Universe generated by $ZFC$. She concludes:

> In sum, then, it seems to me that the familiar set-theoretic foundations, rough and ready as they are, remain the best tool we have

for the various important foundational jobs we want done.(Maddy (2017, p.53))

Her argument mostly relies on UNIFY: the Single Universe still provides a better foundation than the multiverse, whence there is no need to adopt the latter. Now, if one considers Hamkins' broad multiverse (Hamkins (2012)), Maddy's assessment is correct. Because the broad multiverse is so liberal as to accept every possible universe (including e.g non-well-founded universes, trivial universes, and so on), the foundational power of the set theoretic framework is lost. In Hamkins' view, the main job of set theory is to deal with different kinds of constructions, which can verify, or falsify, the same set-theoretic claims (suche as $CH$). In such a framework, there is no reason to ban a particular universe: they are all perfectly legitimate model theoretic constructions, and thus part of the multiverse.

However, Hamkins' conception falls short on exactly this: considering the naturalistic principle UNIFY and the foundationality features that come with it (Meta-mathematical corral, Risk assessment, Elucidation and Shared Standard), his multiverse cannot provide all these features. The main reason is that there is no "bridge" between universes: every universe is isolated from the others, with its particular concept of set and membership. The main consequence is that we cannot define a shared notion of proof, nor prove or define anything shared between all the universes. By contrast, the Single Universe $V$ delivers all these features: we can prove something general about all mathematics in it, since it can all be reduced in one "manageable package" (Meta-mathematical Corral); we can use set-theoretic definitions to clear otherwise ambiguous mathematical objects (Elucidation); it is a very good standard for what counts as a mathematical proof (Shared Standard); and, finally, it is very useful when trying to assess the problems with the use of a certain mathematical object or method. All in all, we can say that ZFC does a very good job as a "foundation" for mathematics.

However, Maddy appears to only focus on UNIFY, and to disregard MAXIMIZE. The main advantage of any pluralistic framework is that it allows for the existence of many more structures and objects than the Single Universe. For instance, in the usual set theoretic universe $V$ we must chose which kind of set theory is actually generating the universe (for example, the $V$ generated from $ZFC$ is radically different from the universe generated by $ZF + AD$), thus restricting the available objects and structures. By contrast, in the multiverse all these different universes would live side by side, ready to be studied and compared *within* the same framework.

This is perfectly in line with the naturalistic principle MAXIMIZE: a theory is preferable over another if it proves more isomorphisms. In other words, our theory should be a "Generous Arena", where we can compare, study and analyze as many objects and structures as possible. Having all these structure in the same framework is very helpful for mathematical practice: one can prove general results about them, search for patterns, and so on.

But how are we sure that the multiverse can actually prove more isomorphisms? To see this, let us consider the most simple case: a multiverse made of two very similar universe, that differ for the least possible amount. In particular, let's assume that one of these universes is actually equal to $V$, and the other differs from it for just one structure. In this case, we can prove that most of the structures of $A$ are isomorphic to the correspondent structure of $A'$. We then get new isomorphisms (in this case particular case, trivial ones) that are unprovable in $V$ alone.

Returning to the more complex (and not trivial) case of a full multiverse, we have shown that our new framework is a more Generous Arena than the Single Universe: there are more mathematical structure and objects than can co-exist in it, and one can study their interrelations.

If the only possible multiverse was the broad multiverse, Maddy's diagnosis would still be correct: the advantages of MAXIMIZE don't counterbalance the disadvantages of the lost foundationality. The naturalist pluralist, then, would need to find a multiverse conception that is powerful enough to satisfy MAXIMIZE while retaining the foundationality of UNIFY. I would like to argue that the generic multiverse with a core $GM_H$ (Steel (2014)) is precisely what the pluralist needs.

In this kind of multiverse, we define a criterion to narrow the possible universes, thus "compacting" the multiverse in a coherent plurality of universes. This criterion is the existence of a core: indeed, the $GM_H$ is built around a common core of set theoretic truths, common to *all universes of the multiverse*. This universes are all extensions of $ZFC$, or, semantically, all the universes of the $GM_H$ are extensions of $V$. Such a construction ensures the satisfaction of both UNIFY and MAXIMIZE.

Indeed, since the $GM_H$ is a multiverse, our argument above for MAXIMIZE still applies: we can prove more isomorphisms in it than in the Single Universe. But how foundational is the $GM_H$? Recall, Maddy's main argument against the multiverse was that it wasn't foundational enough. However, the argument no longer applies in the present case because of the existence of the core. As a matter of fact, since this core is common to *all* universes, we can easily define in it all the foundational feature considered by Maddy. The core assures us that the rules are the same in every universe, so our definition of proof will be preserved across the whole multiverse (Shared Standard). Also, the language and rules we are using will be the same in all the multiverse, so a definition of a mathematical object in one universe will still be understandable in other universes (Elucidation). Moreover, if a mathematical object clashes with something in our core it will be problematic to use it in every universe of the multiverse (Risk Assessment). Finally, we will be able to prove something general about not only mathematics, but also the multiverse itself, since it will be included in the core (Meta-mathematical Corral). Moreover, this core is $ZFC$ itself, so we are sure that the $GM_H$ is as foundational as $V$: everything that is true in $ZFC$, is true in the $GM_H$.

Thus, we can conclude that the $GM_H$ satisfies both MAXIMIZE and UNIFY, so, from a naturalistic point of view, we can only conclude, pace Maddy, that is better than the Single Universe as a framework for mathematical practice.

# 4 Concluding remarks

In conclusion, I have argued that, from a naturalistic point of view, the $GM_H$ is better than the Single Universe as a framework for mathematical practice. This is because is just as foundational as the Single Universe, thus satisfying the principle UNIFY, and it proves more isomorphisms than the Single Universe, thus satisfying MAXIMIZE.

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