Week 8

Solutions to exercises on first order semantics

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1. Consider the following formulas:
   a) $\forall x[P(x)] \leftrightarrow \neg\exists x[\neg P(x)]$;
   b) $\exists y[\forall x[P(x) \rightarrow Q(x, y)]]$;
   c) $\neg\exists x[P(x)] \leftrightarrow \forall x[P(x)]$.

2. Which one is a satisfiable, unsatisfiable and valid?
   a) To evaluate the first formula, the best course of action is to apply the logical consequence definition given in week 8. This means doing the following:
      - $\mathcal{M} \models \forall x[P(x)] \leftrightarrow \neg\exists x[\neg P(x)]$ if and only if $\mathcal{M} \models \forall x[P(x)]$ AND $\mathcal{M} \models \neg\exists x[\neg P(x)]$, OR $\mathcal{M} \not\models \forall x[P(x)]$ AND $\mathcal{M} \not\models \neg\exists x[\neg P(x)]$
      - but from the definition we know that $\mathcal{M} \models \forall x[P(x)]$ if and only if $\mathcal{M}\neg\exists x[\neg P(x)]$, so there is no way that there exists a model $\mathcal{M}$ that satisfies $\forall x[P(x)]$ and not $\neg\exists x[\neg P(x)]$.
      - Thus, this formula is valid (i.e. satisfied by all models).
   b) The evaluation of this formula is easier, you just need to find a model that satisfies it and another model that doesn't. This is satisfiable.
   c) Finally, this last formula is unsatisfiable (to see it, just apply the same reasoning of formula (a), but this time you reach the paradoxical conclusion that a model satisfies the first half if and only if there is something that is not P, but it should satisfied also the second part, in which it is stated that everything is P).

3. There are entailments between these formulas?
   - (a) is entailed by both (b) and (c), since a valid formula is entailed by everything (every model satisfies it, so every model that satisfies another formula also satisfy it).
   - Trivially, (b) is entailed by (c), since every model that satisfies (c) also satisfies (b). This is because it is not possible to find a model that satisfies (c) that does not satisfy (b) (trivially because it is not possible to find a model that does satisfy (c)).