Week 9

Notes on first order logic semantics

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Now that we know how models and the satisfaction relation work, we can expand our model theoretic tools and start talking about the wider *semantics* for first order logic.

As you remember, the semantics of propositional logic was quite simple: we have formulas that are always true, formulas that are always false and formulas that some times are true and some times are false. We also had a simple method to check what a formula was. Last week we have seen that this is could not work for first order logic formulas, and we need something different. That is way we introduced models and the satisfaction relation. In first order logic "true" and "false" do not make too much sense in the same way than propositional logic: a formula in first order logic is not "true" or "false" tout court, but it is satisfied or not satisfied by a particular model (of our choosing). Moreover, that same formula can be satisfied by a model, but there could be easily another model that does not satisfy it! In other words, in first order logic we say that **the truth of a formula is *relative* to a model**.

However, we still want to say something very general about all this formulas, and try to assess them in the same terms of propositional logic, i.e. we still want to know if a formulas is "always true", "always false" or "contingent".

We can start with "contingent" formulas. In propositional logic, a formula is contingent if in some cases it is true, and in other cases it is false. The correspondent semantic notion in first order logic is very similar: if a formula has a model, it is said to be *satisfiable*, even if there is also a model that does not satisfy it. For example, the following formula

$$\forall x[P(x) \to Q(x)]$$

is satisfiable by the model $\mathcal{M}$:

- $\mathcal{D}^{\mathcal{M}}$ = all men;
- $P^{\mathcal{M}}(x)$ means "x is a man";
- $Q^{\mathcal{M}}(x)$ means "x is mortal".

Consequently we can say that this formula is satisfiable, since we were able to find a model for it. And this is still the case even though we can find also a model that does not satisfy it. For example, the model $\mathcal{N}$ does not:

- $\mathcal{D}^{\mathcal{N}}$ = all natural numbers;
- $P^{\mathcal{M}}(x)$ means "x is odd";
- $Q^{\mathcal{M}}(x)$ means "x is prime".

since we can provide a counterexample for that formula (for example the number 9 is odd but it is not prime).

Definition 1 (Satisfiability)**.** A formula φ is *satisfiable* if and only if there exists a model $\mathcal{M}$ such that $\mathcal{M} \models \varphi$.

The opposite of a satisfiable formula is an *unsatisfiable* one. In this case, we cannot find a model that satisfy it, no matter how hard we try. Last week we already saw a formula in first order logic that it is unsatisfiable:

$$\forall x[P(x) \wedge \neg P(x)].$$

It is clear that unsatisfiable formulas are connected to contradictions in propositional logic: usually they have the same logical structure.

Definition 2 (Unsatisfiability)**.** A formula φ is *unsatisfiable* if and only if there is no model $\mathcal{M}$ such that $\mathcal{M} \models \varphi$.

We can also express this definition in another way: a formula is unsatisfiable if and only if every model $\mathcal{M}$ does not satisfy it.

The last semantic notion we need to finish our parallel with propositional logic is the notion that should correspond with the one of tautologies. In the case of first order logic, this notion is the one of *validity*. It is actually really similar to the validity we already encounter at the start of the course, when we talked about arguments. A formula is valid if and only if we cannot find a model that does not satisfy it, just like in the case of arguments (where an argument is valid if and only if we cannot find a way in which the premises are accepted and the conclusion is not). In other words, if a formula is satisfied by every model, then it is valid.

Definition 3 (Validity)**.** A formula φ is *valid* if and only if, for every model $\mathcal{M}$, $\mathcal{M} \models \varphi$.

All the laws of logic are valid in first order logic. So for example the formula

$$\forall x[P(x) \vee \neg P(x)]$$

is valid, since all the models we can think of satisfy it.

To check if a formula is valid, you need to try to build a model that does not satisfy it. Since doing this from scratch can be daunting, the best way to approach is to try first to build a model that satisfies its opposite. So for example if we need to check the formula

$$\forall x[P(x)]$$

we try to build a counterexample, i.e. we try to build a model that satisfies

$$\exists x[\neg P(x)].$$

If this is not possible, then our first formula is valid.

Before concluding this class, we need to talk about another semantic notion, *entailment*. This semantic notion is unique for first order logic, and there is no correspondent notion in propositional logic. A formulas entails another formulas if and only if every time it is satisfied by a model, that same model also satisfies the other formula. For example, suppose we have the following formulas:

1. $\forall x[P(x)]$;
2. $\exists x[P(x)]$.

It is easy to show that every model that satisfies the first one also satisfies the second one. To prove it, we try to build a model that satisfies only the first one, and not the second one. But this cannot be done, just like it is not possible to accept the following sentences together:

1. Every man is mortal.
2. There exists and immortal man.

Definition 4 (Entailment)**.** A formula φ *entails* a formula ψ if and only for every model $\mathcal{M}$ such that $\mathcal{M} \models \varphi$, also $\mathcal{M} \models \psi$ holds.

It is really similar to the notion of implication, if we want to draw such an analogy, but with one important distinction. In propositional logic, if one formula implies another formulas what we have is a third formulas in which the first formulas is the antecedent and the second formulas is the consequent. Moreover, the only way to check such a case is to simply build a truth table for that formula. On the other hand, in the case of first order logic we are using models, and things are a little bit different.