Week 7

Notes on the syntax of first order logic

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Now that we know how propositional logic works, we need to step to the next level, and start studying *first order* logic. In this class I will introduce the language and the syntax of first order logic, so that you can start to understand formulas in this new language. Also, you will see how to translate between natural language and the language of first order logic.

Indeed, while propositional logic is fun (!) and, in its classical form, also quite powerful, it is still not enough for our purposes. Remember that we want to develop a formalisation for our natural language, so that we can assess the truth/validity of statements and arguments without being distracted by the features of natural language. For example, take the following sentence:

All employees are not insured.

How can we translate this in the language of propositional logic? Well, we can use two predicates $I(x)$ that means "x is insured" and $E(x)$ that means "x is an employee" and translate it like this:

$$\text{For all } x, E(x) \to \neg I(x)$$

But this would not be a complete translation, we still have to translate "For all".

This is the reason in first order logic we add two more symbols to the language of propositional logic, called *quantifiers*:

- the *universal quantifier*, "$\forall$", means "every" (equivalent to "for all, each");
- the *existential quantifier*, "$\exists$", means "there exists".

With this two symbols, we can then fully translate our sentence:

$$\forall x[E(x) \to I(x)].$$

To finish the introduction of these two new symbols, we need to update our definition of language (in red the differences with the definition of propositional logic's language given on week 3):

Definition 1 (Language of First Order Logic)**.** The language of first order logic is composed of:

- a *basic alphabet*;
- an *extended alphabet*.

The basic alphabet of first order logic is build with the following symbols:

- an infinite number of symbols for *names*: $a, b, c, \ldots$;
- an infinite number of symbols for *variables*: $x, y, z, \ldots$;
- an infinite number of *unary* (i.e. with just one place in the parenthesis) symbols for *properties* (predicates): $P, R, Q, \ldots$;
- an infinite number of symbols with no restriction of arity (i.e. the number of places in the parenthesis) for *relations*: $T, S, M, \ldots$;

The extended alphabet of first order logic is the basic alphabet plus the following symbols:

- the *logical connectives*: $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication) and $\leftrightarrow$ (biconditional);
- the universal, $\forall$, and existential, $\exists$, quantifiers;
- the *punctuation* symbols: "," (comma), (,) (parenthesis), and "." (point).

Nothing else is part of the language.

As you can see the definition is just the same, with the only meaningful addition of the quantifiers. Also the definition of well formed formulas needs a similar update (the definition of atomic formulas stays the same):

Definition 2 (Well-formed formulas)**.** The set of all *well-formed* formulas of first order logic is formed by the following:

1. All atomic formulas are well-formed formulas;
2. If φ is a well-formed formula, then $\neg\varphi$ is a well-formed formula;
3. If φ and ψ are well-formed formulas, then $\varphi \wedge \psi$ is a well-formed formula;
4. If φ and ψ are well-formed formulas, then $\varphi \vee \psi$ is a well-formed formula;
5. If φ and ψ are well-formed formulas, then $\varphi \rightarrow \psi$ is a well formed formula;
6. If φ and ψ are well-formed formulas, then $\varphi \leftrightarrow \psi$ is a well formed formula;
7. If φ is a well formed formula, then $\forall x[\varphi]$ is a well formed formula;
8. If φ is a well formed formula, then $\exists x[\varphi]$ is a well formed formula;
9. Nothing else is a well formed formula.

Now that we have the full language of propositional logic, we can go on and translate any sentence or argument from the natural language. For example, the classic argument:

(1) Every man is mortal.
(2) Plato is a man.

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(3) Plato is mortal.

can be translated as:

$$\begin{array}{ll} (1) & \forall x[M(x) \rightarrow D(x)] \\ (2) & M(p) \\ \hline (3) & D(p) \end{array}$$

Or, in the case of this invalid argument:

(1) Some Irish are catholic.
(2) Douglas is Irish.

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(3) Douglas is Catholic.

the following would be a faithful translation:

$$\begin{array}{ll} (1) & \exists x[I(x) \rightarrow C(x)] \\ (2) & I(d) \\ \hline (3) & C(d) \end{array}$$

From these two examples we can learn several things about how quantifiers works.
First of all, focus on the first premise of both arguments:

1. $\forall x[M(x) \to D(x)]$;
2. $\exists x[I(x) \to C(x)]$.

These two formulas translate "predicate" sentences, i.e. sentences of the form "Every/Some x are y". But we are translating them in a different way. What happens if we then try to translate them back to natural language? We would end up with this result:

1. For every x, if x is a man then x is mortal.
2. There exists x such that if x is Irish then x is catholic.

Why are we translating those sentences in this conditional form? The reason behind this is the *range* of the quantifiers. Every quantifier has a precise range: the *domain* of our discourse. This domain encompasses all the things that we are talking about. For example, if we are talking about flowers, then the domain of our discourse would be all the species of lowers, and maybe also the single flowers. If we are talking of the balls in a ball pit, then the balls of that particular pit are the domain of our discourse. If I say in front of a pit ball "All the balls are red" then you could check in that particular pit ball if what I said is true. When doing logic, we cannot use any contextual clue to limit our domain of discourse, so we need to use other methods. If I say "$\forall x[R(x)]$" in logic I simply mean "Every thing is red". This is quite a general and bold (and easily refutable) claim! Consequently I need to restrict my domain in other ways. What if I use the conjunction? Then, still using our ball pit case I would say:

$$\forall x[B(x) \land R(x)].$$

But this means "Every thing is a ball and red". An even more general and more bold claim! On the other hand, the conditional form restrict our domain of discourse in the right way:

$$\forall x[B(x) \to R(x)]$$

means "For every thing, if that thing is a ball, then it is red". While we need to further restrict our domain to include only the ball of our pit, we are on the right track: we can add all the restrict condition to the left of our conditional and we have successfully restricted our domain:

$$\forall x[(B(x) \land P(x)) \to R(x)],$$

that can be translated as "For every thing, if it is a ball and it is in this pit, then is red" (or, more fluently, "Every thing that it is a ball in this pit is red.")

The other thing we can learn from the two arguments above is how to substitute variables with constants. Consider our situation with the ball pit, and imagine I say:

$$\forall x[(B(x) \land P(x)) \to R(x)],$$

meaning that every ball in the pit is red. Also suppose for simplicity that I am really poor, and that in my pit there are only 3 balls: 2 red ones (balls a and b) and an orange one (ball c). To check if my claim is correct, you need to substitute the variable in my formula with the elements of the domain. In our case, the elements of the domain are 3. Since the quantifier we are using is universal, we need to check *all* the elements of the domain. This means checking if all of the following are true:

$$(B(a) \land P(a)) \to R(a)$$
$$(B(b) \land P(b)) \to R(b)$$
$$(B(c) \land P(c)) \to R(c).$$

Since we said that the ball c is orange and not red, the last formula is not true, and thus the whole claim is false.

If instead the claim was existential: "There exists a ball in this pit that is red":

$$\exists x[(B(x) \land P(x)) \to R(x)],$$

we would need to check only *one* element of the domain, and thus substituting our variable with one constant. For example:

$$(B(b) \wedge P(b)) \rightarrow R(b).$$

And this would also validate our claim.

The last important notion we are going to discuss today is the *scope* of the quantifiers. Do not confuse scope and range: while the range refers to the domain of discourse, the scope of the quantifier refers to the formula. In other words, the range is a semantic notion (in the next two weeks we will discuss semantics in greater detail), while the scope is a syntactic notion.

Consider the following sentence:

Every man lives in a house.

Notice that this sentence is ambiguous: it can mean two very different things:

1. Every man lives in his house.
2. There is a particular house in which all the men live.

In natural language this type of sentence is quite common. Context and other clues help us deciding which interpretation is the correct one. In logic however we cannot count on the context, thus we need to choose which one is the interpretation intended. In this case, the two interpretation have very different formalisations:

1. $\forall x[\exists y[M(x) \rightarrow L(x, y)]]$;
2. $\exists y[\forall x[M(x) \rightarrow L(x, y)]]$.

While the first translation is the right one, the second one is problematic. What I am saying is that for every man in the world, there exists a *particular* house in which he lives. That is, all the men in the world live in only one house (a nightmare in these times of social distancing)!

The scope of a quantifier is the "influence" of the quantifier on the formula. We denote this using square brackets, and it is the only way to translate natural language sentences that are ambiguous. **Every quantifier influence only the nearest complete expression to the right.** For example in the following formula

$$\forall x P(x) \rightarrow Q(x).$$

the scope of the quantifier is only $P(x)$, the x in $Q(x)$ is not influenced by that quantifier. In this case, we write:

$$\forall x[P(x)] \rightarrow Q(x).$$

If we want to extend the scope of that quantifier to the whole formula, we need to write:

$$\forall x[P(x) \rightarrow Q(x)].$$

We call the variables influenced by a quantifier *bound variables*, while the ones that are not influenced by a quantifier *free variables*.

This is particular important in the case of ambiguous sentences. In these cases, we need to use square brackets and the quantifier order to correctly indicate the quantifiers' scope and thus the correct interpretation.