Week 4

Notes on the semantics of proposition logic

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Now that we have looked at how the syntax of the propositional logic works, we move to the semantics. This argument is not very extensive for propositional logic, but it will become much more important for first order logic. The main difference between syntax and semantics in logic is just the same as in linguistics: syntax is interested on the manipulation of the symbols disregarding their meaning, while semantics looks especially to the meaning and interpretation of the symbols. While this is not very important for propositional logic, there are some definitions that are quite special. All of the following ties with the truth tables we defined last week.

Consider the following formulas:

$$p \to (p \lor q)$$
$$p \to (p \land q)$$
$$(p \lor q) \to p$$

They all look very similar. However, the first one is actually special, and stands out. Lets consider their truth table:

p	q	$p \lor q$	$p \to (p \lor q)$
1	1	1	1
1	0	1	1
0	1	1	1
0	0	0	1

p	q	$p \land q$	$p \to (p \land q)$
1	1	1	1
1	0	0	0
0	1	0	1
0	0	0	1

p	q	$p \lor q$	$(p \lor q) \to p$
1	1	1	1
1	0	1	1
0	1	1	0
0	0	0	1

Now the reason is much clearer: the first one is actually always true! A formula that is always true, no matter the truth values of its components, is called a *tautology*.

Definition 1 (Tautology)**.** A *tautology* is a formula that it is always true, no matter the truth value assigned to its components.

Tautologies are an important part of logic. While it may seem that they do not carry meaningful content (for example, a tautology in natural language is "it is what it is"), they are in fact fundamental in defining which kind of logic we are in. Indeed, the most notable tautologies have names and are considered *laws of logic*. For example:

Law of the Excluded Middle $p \lor \neg p$;

Law of Contraposition $(p \to q) \leftrightarrow (\neg q \to \neg p)$;

Reductio ad absurdum $((\neg p \to q) \land (\neg p \to \neg b))\neg p$;

These laws are what define *classical* propositional logic. Intuitionistic logic, for example, does not include the law of the excluded middle. Last week we have seen that it is possible to translate from a logic with only some of the connectives (for example $\neg, \lor$) to another with other connectives (for example $\neg, \land$). These equivalences can be written as tautologies in propositional logic, and are instances (special cases) of what is called *De Morgan's Law*:

De Morgan's Law $\neg(p \land q) \leftrightarrow (\neg p \lor \neg q)$

From a given tautology it is always possible to build more tautologies, substituting its components. For example, take again our first tautology:

$$p \to (p \vee q).$$

We can substitute p with $p \wedge r$ and q with $q \wedge s$ and still get a tautology:

$$(p \wedge r) \to ((p \wedge r) \vee (q \wedge s)).$$

Obviously, this is because no matter how many substitution we make, all the different formulas share the same structure. A tautology that it is not derived from another through this substitution is called a *minimal* tautology. For example, $p \to p$ is a minimal tautology.

The opposite case of a tautology is a formula that it is always false. Such a formula is called a *contradiction*.

Definition 2 (Contradiction)**.** A *contradiction* is formula that it is always false, no matter the truth value assigned to its components.

Considering how connectives works, putting a negation in front of a tautology always produce a contradiction. For example, $\neg(p \to (p \vee q))$ is a contradiction. For a less trivial example, consider the following formula:

$$(p \wedge \neg q) \leftrightarrow (p \to q).$$

In this case, no matter how much we can try, we cannot find a combination of truth values for p and q such that the whole formula comes out true:

p	q	$\neg q$	$p \wedge \neg q$	$p \to q$	$(p \wedge \neg q) \leftrightarrow (p \to q)$
1	1	0	0	1	0
1	0	1	1	0	0
0	1	0	0	1	0
0	0	1	0	1	0

The last case is when a formula is in some cases true and in some other cases false. In our first example of the tautology, the other two formulas weren't tautologies because in one case they were false:

p	q	$p \wedge q$	$p \to (p \wedge q)$
1	1	1	1
1	0	0	0
0	1	0	1
0	0	0	1

p	q	$p \vee q$	$(p \vee q) \to p$
1	1	1	1
1	0	1	1
0	1	1	0
0	0	0	1

If this is the case, then we call that formula *contingent*.

Definition 3 (Contingent formula)**.** A formula is *contingent* if and only if it is neither a tautology nor a contradiction.

Pages 104-105 of the Open Logic Text give the more formal definitions for all these notions. However, they are not needed for this course, and understanding this notes is more than enough.