Week 3

Notes on truth tables

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19 March 2020

This is the second part of today's notes. In the following notes, I will introduce the truth tables for the logical connectives of the language of propositional logic.

The truth tables explain the *meaning* of the logical connectives in such a way that this meaning is rooted in how those connective behave in natural language, but their evaluation then can become independent from it. This means that from now on when evaluating if a formula is true/false we won't need anymore to refer back to the sentence in natural language or how the world actually is, but we can evaluate it in isolation. At this point our trip from natural language to completely formal symbolic logic is complete.

The propositional logic we are learning in this course is *classical*: this means that there are only *two* truth values, true (that we will denote with 1) and false (that we will denote with 0). These two truth value are *assigned* to atomic formulas and sentences: for example if the atomic formula p is true we write $v(p) = 1$. Such assignment is *compositional*, that is the truth assignment of complex formulas is based on the truth assignments of their parts. So for example we can say that the truth assignment of an implication is based on the truth assignment of the antecedent and the truth assignment of the consequent.

Now lets start with the truth table of *negation*. The meaning of negation is to negate what is stated in a formula, so it works like switch: if a formula is true, then its negation is false, and vice versa:

p	$\neg p$
1	0
0	1

The second connective, the *conjunction*, is true only in the case where both the parts of the conjunction are true. It is false in all other cases:

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

In logic, the *disjunction* is always *inclusive*: to be true it suffices that *one* of the disjuncts are true, but they can also be *both* true:

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

This is different than the disjunction in most natural languages, where the common usage is usually *exclusive*. In this case, the exclusive disjunction is true if one of the disjuncts is true, but false if they are both true:

p	q	$p\bar{\vee}q$
1	1	0
1	0	1
0	1	1
0	0	0

However, please note that this exclusive disjunction *is not part of classical logic.* The main reason behind this is that it is not needed, since we can easily achieve the same truth tables just adding a conjunction:

$$(p \vee q) \wedge \neg(p \wedge q)$$

(you can try to check it, if you want).

The *implication* in logic and its truth table is one of the biggest debate in the philosophy of logic and philosophy of language. In logic the implication used is the *material conditional*, since it is the one used in mathematics. in natural language there are various possible different kinds of implication, for example the counter factual conditional ("If my grandmother had wheels, she would have been a car."). However, these other conditionals are not part of classical propositional logic, and are thus not formalise in any connective nor in the truth table of the material conditional. The logical implication is always true, other than in the case where the antecedent is true and the consequent is false:

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

The last connective of propositional logic is the *biconditional.* This connective is true when both part of the formula have the same truth value:

p	q	$p \leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

An interesting property of this set of logical connectives is that is actually *redundant*, i.e. we don't need so many connectives. Indeed, we can have a complete propositional logic with just two connectives: $\neg$ and $\wedge$. This is because all the other connectives can be defined in terms of those two (just like the exclusive disjunction in terms of inclusive disjunction, negation and conjunction). For example, we can define (in this context, having the same truth table) the disjunction as follows:

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg(\neg p \wedge \neg q)$
1	1	0	0	0	1
1	0	0	1	0	1
0	1	1	0	0	1
0	0	1	1	1	0

Just in the same way we can also define implication and biconditional. Moreover, this is true of any pair of connectives: we can also define everything else using $\neg, \vee$ or $\neg, \to$ or $\neg, \leftrightarrow$. This is not done only because the formulas would become quite long and complex, and more difficult to interpret.