Week 11

Notes on natural deduction for first order logic

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In last week notes, I introduced the natural deduction rules for quantifier, thus adapting our classical propositional logic natural deduction system to first order logic. Today, for the last notes of the course, I will show you some more examples of natural deductions, without introducing anything else new. I will use both De Morgan's Law for quantifiers as examples for derivations. This means that in this notes I will give you the derivations of the following formulas:

1. $\exists x[\neg P(x)] \leftrightarrow \neg\forall x[P(x)]$;
2. $\forall x[P(x)] \leftrightarrow \neg\exists[\neg P(x)]$.

Since we didn't introduced any rule for the biconditional, we cannot prove them directly. Instead, we need to prove both directions separately, i.e. I will prove:

1. $\vdash_{FOL} \exists x[\neg P(x)] \to \neg\forall x[P(x)]$;
2. $\vdash_{FOL} \forall x[P(x)] \to \neg\exists[\neg P(x)]$.
3. $\vdash_{FOL} \neg\forall x[P(x)] \to \exists x[\neg P(x)]$;
4. $\vdash_{FOL} \neg\exists[\neg P(x)] \to \forall x[P(x)]$.

Remember from last week that when dealing with quantifiers, we have to make sure not to violate the variables restrictions, and sometimes this requires us to play around with the order of carrying out certain inferences. In general, it helps to try and take care of rules subject to the variable restrictions first (they will be lower down in the finished proof).

First of all let's focus on the first of the De Morgan's Law, $\exists x[\neg P(x)] \leftrightarrow \neg\forall x[P(x)]$. Since there are no rules for the biconditional, we need to prove to each direction. First of all, we prove $\vdash_{FOL} \exists[\neg P(x)] \to \neg\forall x[P(x)]$.

$$\dfrac{\dfrac{[\exists x[\neg P(x)]]^1 \qquad \dfrac{\dfrac{[\neg P(a)]^2 \qquad \dfrac{[\forall x[P(x)]]^3}{P(a)}\langle\forall\mathrm{E}\rangle}{\bot}\langle\neg\mathrm{E}\rangle}{\neg\forall x[P(x)]}\langle\neg\mathrm{I}\rangle,3}{\neg\forall x[P(x)]}\langle\exists\mathrm{E}\rangle,2}{\exists x[\neg P(x)] \to \neg\forall x[P(x)]}\langle\to\mathrm{I}\rangle,1$$

It is important, especially when dealing with quantifiers, to double check at this point that the variable restrictions has not been violated. Since the only rule we applied that is subject to the variable restrictions was Existential Elimination, and the constant a does not occur in any assumptions it depends on, this is a correct derivation.

The easiest way to prove the other direction, i.e. $\vdash_{FOL} \neg\forall x[P(x)] \to \exists x[\neg P(x)]$, is to assume both $\neg x[\neg P(x)]$ and $\neg P(a)$, so we can apply the Classic reduction rule (for two times). Note that we are proving $\neg\forall x[P(x)] \to \exists x[\neg P(x)]$ from no assumptions, so this means that all the assumptions we make to prove it have to be discharged.

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\neg\forall x[P(x)]]^1 \qquad \dfrac{\dfrac{\dfrac{[\neg\exists x[\neg P(x)]]^2 \qquad \dfrac{[\neg P(a)]^3}{\exists x[\neg P(x)]}\langle\exists\text{I}\rangle}{\bot}\langle\neg\text{E}\rangle}{P(a)}\langle\text{CR}\rangle,3}{\forall x[P(x)]}\langle\forall\text{I}\rangle}{\bot}\langle\neg\text{E}\rangle}{\exists x[\neg P(x)]}\langle\text{CR}\rangle,2}{\neg\forall x[P(x)] \rightarrow \exists x[\neg P(x)]}\langle\rightarrow\text{I}\rangle,1}{}
$$

Again, we need to check that there are no problems with our variables. The only constant a is not a problem, since it is already a discharged assumption when we apply the Universal Introduction. Thus this is a good derivation, and we do not need to worry.

What about the other De Morgan's Law? We know that it asserts the equivalence between a universal quantifier and a negated existential quantifier: $\forall x[P(x)] \leftrightarrow \neg\exists[\neg P(x)]$. We start by proving one side of the law, i.e. $\vdash_{FOL} \forall x[P(x)] \rightarrow \neg\exists[\neg P(x)]$. As before,this is because in natural deduction there are no rules for the biconditional, so we need to prove both direction separately. To do the first one, we assume $\neg existsx[\neg P(x)]$ to apply $\langle\neg\text{I}\rangle$, and then $\neg P(a)$ for the application of $\langle\exists\text{E}\rangle$.

$$
\dfrac{\dfrac{\dfrac{[\exists x[P(x)]]^2 \qquad \dfrac{[\neg P(a)]^3 \qquad \dfrac{[\forall x[P(x)]]^1}{P(a)}\langle\neg\text{E}\rangle}{\bot}\langle\neg\text{E}\rangle}{\bot}\langle\exists\text{E}\rangle,3}{\neg\exists[\neg P(x)]}\langle\neg\text{I}\rangle,2}{\forall x[P(x)] \rightarrow \neg\exists[\neg P(x)]}\langle\rightarrow\text{I}\rangle,1
$$

Here we do not have any conflict, since there are no constant in common between $\exists x[P(x)]$ and other undischarged assumptions.

To prove the other side, i.e. $\vdash_{FOL} \neg\exists[\neg P(x)] \rightarrow \forall x[P(x)]$, we only need to assume $\neg P(a)$ so that we can apply Classical Reduction.

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\neg\exists x[\neg P(x)]]^1 \qquad \dfrac{[\neg[P(a)]]^2}{\exists x[\neg P(x)]}\langle\exists\text{I}\rangle}{\bot}\langle\neg\text{E}\rangle}{P(a)}\langle CR\rangle,2}{\forall x[P(x)]}\langle\forall\text{I}\rangle}{\neg\exists x[\neg P(x)] \rightarrow \forall x[P(x)]}\langle\rightarrow\text{I}\rangle,1
$$

Finally, in this last derivation also we do not need to worry about constant and variables conflicts. Indeed, the only application of a potential problematic rule, the Universal Introduction before the last step, comes when the constant a in the first assumption is already discharged.