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"Middle Earths"

The existence of alternative universes in mathematics and the Continuum Hypothesis

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Indice

Introduction

In this introduction I will discuss, at a very general level, of pluralism and its mathematical consequences: after summarizing the two main pluralist positions I will propose a more detailed classification. As we will see in the following chapters, one of the issues that differentiate pluralists and anti-pluralists is the justification of new axioms. For a pluralist is always possible to introduce new axioms, justifying them only on practical bases (for example, to see what universes is generated from a specific axiom system). Instead, for an advocate of the anti-pluralism all the new axioms must be justified on a theoretical level. In fact the majority of the problems are based on the limits of ZFC, and a lot of these limits were made evident by the independence results.

On Pluralism and Anti-Pluralism

The fundamental idea of pluralism is that we do not have enough evidence to believe that mathematics is a single monolithic truth and even to think that the truth in the whole mathematics is reducible to the truth in a particular mathematical theory, which will be the foundational theory of mathematics. The pluralist is therefore an agnostic, but not one that thinks that we should stop studying philosophy of mathematics. On the contrary, his agnosticism is the first motivation to continue the research. We can characterize such a position as a *not so radical* skepticism: the pluralist considers mathematics a number of theories, each one containing a relative truth. Or the pluralist may regard mathematics as a process rather than a block of monolithic truth. In both cases, the core of pluralistic thought is the presence of more than one notion of truth: in the first case, each single theory will have its particular truth. In the second the concept of truth will be tied to a certain time of the process: a different time will be linked to a different truth (for example, the Euclid's notion of truth will be different from that of Tartaglia).

In practice, this means accepting the presence of more than one foundational theory. Moreover, all these theories are to be considered at the same level, i.e. equally preferable. Of course they cannot be chosen together, as there may be some contradictions that make trivial the foundational theory.

Take for example the case of ZF as a foundational theory: we can extend ZF with the addition of new axioms and thus have a ZF extensions. For example, we can have ZFC, as well as $ZF + \neg C$, or $ZF + AD$. These extensions, for the pluralistic in set theory, are to be considered, at least in principle, completely equivalent. Of course, historically the preference went to ZFC, but this does not mean anything for the pluralist if for his research field is better to take $ZF + AD$, the pluralist will have no problems or qualms in doing so. Notice how these extensions are not compatible with each other: for example, you cannot consider $ZFC + AD$.

Finally, the pluralist solve without too many problems even all the independent questions from the theory that took into account. Simply, if a statement is found to be independent from a theory, the pluralist states that the proposition is no longer "open", but was settled by showing that *there is no answer within that theory.* Of course, you can always consider a new theory, in which the independent clause (or its negation) will be taken as an axiom and thus resolve the issue. For example, Cohen defended this position on the CH.

On the other hand, anti-pluralism is exactly the opposite belief: mathematics is ruled by a single notion of truth, reducible to one particular mathematical theory, its "foundation". This foundational theory is an axiomatic theory in which one can express and reduce all existing mathematics. In addition, while pluralism does not imply any kind of ontology (i.e., does not say anything on *what* are the objects that studies), for the anti-pluralistic the ontology of the foundational theory *is* the ontology of mathematics: in other words, mathematics studies the objects of its foundational theory.

This is quite obvious if we take as a foundational theory, from an anti-pluralist perspective, ZFC. The truth in *all* math will be the truth in ZFC, and also mathematics is the study of sets, and only sets. Of course, in other areas of mathematics one can speak of other objects (for example, fields), but these will be actually sets, and we refer to them as sets not only for practical convenience.

The Multiverse and the Universe

The concept of "multiverse" was born, in mathematics, following the discovery of the phenomenon of independence in set theory: set theory propositions (e.g. CH) turned to be independent from the axioms of ZFC. To prove it, were used *models* (universes) different from the canonical one: the collection of all these models (universes) constitutes the multiverse.

The multiverse then consists of all the models that satisfy the axioms. In addition, these models contain all the relevant information (although sometimes mutually alternative) on sets. Each of these models is a legitimate universe of set theory, so there is no Single universe. This lack of unity cannot be repaired in any way: set theory is precisely the study of these alternative universes, in which the properties of sets can vary greatly from one to another. Such a view is therefore compatible with pluralism: each of the individual models (universes) will exemplify a different foundational theory, ruled by a different notion of truth.

On the other hand the concept of an unique universe is typical of an anti-pluralist vision: there is only a single structure of set theory featuring all the properties of the sets. The fact that the current axioms of ZFC are characterized by other possible structures simply implies that the axioms taken into account are not enough to describe the universe. From this point of view, it will always be possible to take new axioms to reduce the indeterminacy of set theory and to give an image more and more precise and defined of the universe. They are however aware that a goal like that can never be fully achieved, and our understanding of the universe of set theory will always be partial.

Considering the current developments in set theory, both positions are defensible, especially considering that both undergo some problems. For what concerns the multiverse, the problem is how to justify, taking into account the existence of multiple structures, they can all be seen as "inside" V (in fact, as far as all these structures can be mutually incompatible, they must however be all compatible with V). Instead the only problem of the universe is justify the epistemic relevance of these alternative structures, which are considered as not definitive, but still be able to provide "true" information on sets (for example, in the case of independence demonstrations).

A proposed classification of all the possible positins

Obviously there are many shades of these two positions. One possible method of classification is to consider a new criterion of differentiation, their *realism*. In this way, we can classify them according to their commitment to the *objective* existence of the universe or the multiverse. The realism we are talking about is the *ontological* realism, and not the one about truth values.[1] We can then further divide the universe into two positions and the same can be done with the multiverse, thus getting four possible concepts:

- the *real* universe, similar to Gödel's Platonism;
- the *anti-realist* universe, similar to Maddy's "light realism";
- the *real* multiverse, similar to Balaguer's fullblooded Platonism;
- the *anti-realist* multiverse, typical of Shelah.

The first position, the realist universe, is typical of a Platonist that shares the thought of Gödel. We can sum up this position with the following quotation from Gödel himself:

[1] See [Shapiro 2000] and [Shapiro 2005] for a discussion of the difference between the two types of realism.

> It should be noted, however, that in accordance with the point of view adopted here [that of Platonism], a proof of the independence of Cantor's conjecture from the axioms of the accepted theory of sets (unlike, for example, a proof about the transcendence of π) would not solve the problem at all. Indeed, if the meaning of the primitive terms of set theory [...] is accepted as correct, it follows that the concepts and theorems of set theory describe a definite reality in which Cantor's conjecture must either be true or be false . [2]

According to this view, there is a set-theoretic reality where every proposition has a determined truth value. When studiyng set theory, we refer to this separate reality, and each proved theorem allows you to "discover" new truths of this reality.

But you can defend the universe position without engaging in its actual existence. For example, you can consider the universe as "almost" confirmed by some mathematical results, without trying to justify it from an ontological point of view. For example, Maddy says that the universe should be as large as possible in order to produce a unified area where you can practice all of mathematics, without having to resort to extensions.[3]

A possible intermediate position, a *moderate* realism, was described by Putnam in [Putnam 1979]. A defender of this position will not engage in the actual ontological existence of a single universe, but still believes to be able to find evidence of its existence. We can therefore characterize a moderate realist as a realist that suspends his judgment until the discovery of sufficient evidence to engage in complete Platonism.

The realist multiverse is probably the most peculiar of all these conceptions: those who believe in the real multiverse believes in the "platonic" existance of each of the universes that make up the multiverse, and those universes correspond to different conceptions of set. Such a position has been clarified and developed, from the philosophical point of view, by Balaguer (cfr. [Balaguer 1995] and [Balaguer 1998]), while its more "technical" proponent is Hamkins.

Finally, the last position being considered is the one for which the universes do not exist in a platonic way, they are merely a practical phenomenon that emerges in the study of set theory. This position is the one that requires less philosophical justification of all, as it not only denies the existence of the multiverse, but also of the universe itself. For the defense of this position is normal to become a formalist (is, for example, the case of Shelah).

[2] From [Gödel 1947]. For a discussion on the evolution of Gödel's thought, see [van Atten & Kennedy 2003], [Wang 1974] and [Wang 1996].

[3] See [Maddy 1997] and [Maddy 2011].

On the introduction of new axioms and the status of universes

One of the breaking point (other than the existance of more than one universe) between pluralist and anti-pluralist is the justification of new axioms: for the former, it is always possible to introduce new ones, based purely on practical reasons (for example, to see what universe arises from a certain set of axioms instead from another). Instead, for an anti-pluralist introducing new axioms requires a deep justification in the theory considered (for example, the new axioms must satisfy the definition of MAXIMIZE and UNIFY from [Maddy 1998], so they cannot be introduced at will). This quest for new axioms has a deep relation with the independence results in set theory. In fact most of the problems arise from the intrinsic limits of ZFC, many of which evident from the results of independence.

The most problematic results for set theory are those of independence. According to the pluralist position these results, as limiting the foundational strength of set theory, only undermine the claim of set theory to be a *realist* theory. For the pluralist there are no *theoretical* reasons to prefer a group of axioms over another, but only *practical* motivations. The main consequence of this is that there is not an *objective* mathematics, but many mathematics, one for every purpose of research that is pursued. This position, generally anti-Platonic, leads to the development of the so-called *multiverse*: there is not an unique universe that contains all possible mathematics (i.e. the cumulative hierarchy), but many universes, which differ among themselves for the axioms on which they are based and the truth value of independent issues (e.g. the Continuum Hypothesis). The situation is similar to that of geometry, but with an important difference: while it is possible to consider (naively) the Euclidean geometry more "real" than the non-Euclidean ones, because it is based on the real world and "perceivable", instead in set theory is not possible to prefer one universe to another on the basis of "real world based" arguments. In fact, in the multiverse a certain universe may be preferred over another for its utility for certain purposes, but it is not possible to consider one universe a more "real world" than others. From this perspective, the multiverse is very similar to the semantics of modal logic with possible worlds: a sentence is true if and only if it is true in every possible universe, and it is false if it is false in every possible universe (it is indeterminate if in some it is true and in others it is false).

Parte I

The Multiverse and the Universe

Capitolo 1

The Broad Multiverse

In this chapter I will discuss of the broad multiverse. This position is the most radical among the pluralistic ones: according to this point of view, each consisting theory is part of the multiverse. Obviously this leads to having to solve many problems, including that of the choice of a foundational theory that fits. But this position is so radical that invest the foundational theory itself, and so on. The only way to avoid this problem is to adopt a position less radical and more moderate: every consistent theory, *except* a core that will serve as the foundational theory, is part of the multiverse. This leads to narrow the multiverse models using a particularly strong logic. There are several ways to arrive at a formulation suitable for this multiverse:

- by adding structures to the ontology of the multiverse's theories (like in Hamkins' Multiverse);
- considering the individual universes as mere "labels" (there are several champions of this view, the most notable are Shelah and Foreman) ;

In chapter 2 instead I will consider the *generic* multiverse. This position is particularly suitable in considering universes in which axioms for large cardinals and definable determinacy have a fixed truth value (in the sense already specified), while sentences like CH do not. As with the broad multiverse, there isn't only one way to formulate this position: it is possible to consider the concept of truth in the whole multiverse reducible to the truth in a fragment of the multiverse (chapter 2.2), or to state the exact opposite (chapter 2.1).

As we have already said in the Introduction, there is not a single Pluralism. In fact, is very difficult for a mathematician to be a *pure* pluralist: it's a lot more probable that, in some areas of mathematics, our mathematician is an anti-pluralist. For example, she can be a pluralist in regards of set theory, but an anti-pluralist in regards of PA (in this case we can maybe talk about a finitist). Or we can have a quite opposite situation: an anti-pluralist in regards of the core of ZFC that is a pluralist regarding large cardinals axioms and Continuum Hypothesis. Needless to say, in either cases our mathematician is a pluralist. So we can divide the pluralist position in more moderate and more extreme ones.

The most radical of these positions is the one of the *broad multiverse*. This position take into consideration every area of mathematics: every consistent theory could be a legitimate candidates and the models of these theories are, obviously, legitimate candidates to be models of all the areas of mathematics. Thus everyone of these models is a distinct mathematical universe. But there are some problems with this conception. First of all, we need a background theory where we can discuss the various models that we take into consideration. This is fundamental to avoid trivial consequences[1]. The background theory has to be consistent and stronger than the theory of which the models will be models of. For example, if the theory T is $ZFC + Con(ZFC)$, then the multiverse will be composed by all the models (universes) that satisfy ZFC. Moreover, to avoid triviality, we should be capable of proving the consistency of the theories that we took into consideration. In this background theory we will consider the theory $\mathfrak{T}$, from which we want to derive some models (universes) for the broad multiverse. Now, assuming that $\mathfrak{T}$ is consistent, we know from the second incompleteness theorem that $\mathfrak{T}$ cannot prove its own consistency. So we can find in the broad multiverse also models of $\mathfrak{T} + \neg Con(\mathfrak{T})$. This models are, for the radical pluralist, legitimate candidates, and so universes of the broad multiverse. But, as said earlier, we can get to this conclusion only assuming the consistency of $\mathfrak{T}$, since it is a prerequisite to the application of the second incompleteness theorem. Yet, this is a paradox: from the prospective of the background theory the models we are studying satisfy a false proposition, that is $\neg Con(\mathfrak{T})$. In other words, we lack armony between theory and metatheory. An example will clearify this point. Lets take as background theory primitive recursive arithmetic (PRA) with transfinite induction up to the ordinal ε_0[2] and use it to study Peano arithmetic. In the broad multiverse, since PA cannot prove $Con(PA)$ (by the second incompleteness theorem, assuming the consistency of PA), there will be models of $PA + \neg Con(PA)$. These models, obviously, are legitamate candidates to be mathematical domains in the broader sense, that is universes in the broad multiverse. But this conclusion requires that, in the background theory, we can prove $Con(PA)$, otherwise we cannot apply the second incompleteness theorem as done. Thus, the conclusion is paradoxical: our background theory proves $Con(PA)$, while the theory studied can generate models (and so universes) from $\neg Con(PA)$, that we know it's false. The foundations of our multiverse are shaken: in fact we can find in it universes that, by our background theory, cannot exists. It's as

[1] Thus a better name for this conception could be *broad multiverse relatively* to the theory T, where T is the background theory chosen.

[2] This system was used by Gentzen to prove PA consistency.

strange as a string theorist that studies universes where the thermodynamic laws are false!

The only way to overcome this difficulty is considering every possible configuration of the multiverse as provisional, thus taking a a pluralist stance *also* in regards of the background theory. Then we would have a multiverse of multiverses (one multiverse for every background theory temporarily chosen. But this multiverse of multiverse would need a background theory, with all the difficulties already described. So we would end with a mutliverse of multiverses of multiverses, and so on. But a position like this cannot be defended, and it is very difficult to express in its entirety, resulting in the impossibility to do any mathemtics at all. Therefore even the most radical between the pluralists will limit her pluralism to the theory studied, chosing just one background theory. Only if interrogated and questioned about it this pluralist will shift his pluralism to the background theory, hence taking a firther step back, and so on. In other words, a radical pluralist will always shift focus from a background theory to another.

Now lets consider some positions more tenable. To do that, we have to limit the scope of our research to the positions inside a foundational frame. First of all lets limit the multiverse to set theory (this is purely arbitrary, we can still talk about multiverses of analysis, algebra and so on). Now we have an open sequence of extensions of ZFC based on the iteration of the operation of consistency. This sequence is determinated, so we can use any element of this sequence, lets say $ZFC + Con(ZFC)$, as background theory. With this approach we can use a stronger theory ($ZFC + Con(ZFC)$) as background theory of a weaker theory (ZFC), so the multiverse will be the collection of all the models that satisy ZFC (the weaker theory). We call a multiverse so constructed a *relatively broad multiverse*. This strategy is very useful since allows us to capture the idea that ZFC is determinated and fixed while some proposition, like the CH, are not. In fact ZFC is the same for the whole multiverse, is fixed, while there are universes in which the CH is true and other where is false. The only problem is that *too many* proposition are indeterminated with this approach (e.g., in addition to the CH also Rosser proposition is indeterminated). But, if we consider this problem from the prospective of the background theory, we find out that many of these proposition are determinated! Hence we reach the same paradoxical conclusion: theory and metatheory are mutually inconsistent.

This inconsistency beween theory and metatheory is not our only problem. Assuming a decent articulation of the multiverse conception (such that we can avoid the inconsistency and the infinite regress to solve it) is possible, we have to consider the truth value of certain propositions. In fact, in both the multiverse conceptions (broad and relatively broad one) a proposition has a determined truth value if and only if it has the same truth value in *every* universe of the multiverse. A proposition is true iff it's true in every universes, and it's false iff it's false in every universes. If in some universes is true and in some other is false, then it's indeterminated. The problem is that we can build models (universes) from proposition in contradiction with the background theory. For example:

1. Assuming that ZFC is consistent, it's possible to built not well-founded models that satisfy the proposition "every set is well founded";

2. Always assuming that ZFC is consistent, it's possible to build countable models of ZFC that satisfy the proposition "there exist uncountable sets" although the models themselves are countable;

3. Assuming that PA is consistent, we can build, as already discussed, models of $PA + \neg Con(PA)$;

4. Assuming that Q is consistent, we can build models of $Q + \neg Exp$.

At this point we have to decide how far we can go with the construction of these models (universes). Skolem and von Neumann, for example, accept only cases (1) and (2), stating that the propositions about foundness and countability have no determinated truth value. Thus, for them, only the universes that satisy ZFC and the ones generate from cases (1) and (2) can be part of the multiverse. But any position on this topic is arbitrary, so we can also go all the way down. Lets say that the only property that we consider important to decide if a universe is part of the multiverse or not is its existence. Then every proposition has an indeterminated truth value! This is exactly the position of the broad multiverse: choosing different background theories we can change the truth values of every proposition, thus generating every possible model (universe). Instead the relatively broad multiverse tries to distinguish between the cases where the existence of incompatible models is sufficient to affirm the indeterminateness and the cases where this is not the case. So in this case the simple existence of a model is not a sufficient property to decide if a model is part fo the multiverse or not, hence we need another property. In other words, while in the broad multiverse every existing universes, even if incompatible with the background theory, is part of multiverse by its own right, in the relatively broad multiverse instead we have to find a property to exclude the incompatible models (universes).

1.1 Hamkins' realist multiverse

Hamkins' position[3] can be described as a *radical pluralism*. In fact he is an advocate of the broad multiverse, although it is not clear if he prefers the broad multiverse (more probable) or the relatively broad multiverse (less probable). To be more precise, Hamkins' position is an *implicationism refined by skolemism*. For the implicationist, mathematics consists in drawing

[3] See [Hamkins 2012] and [Koellner 2013] for an extensive comment.

logic consequences from arbitrary chosen axioms. It is not required that these axioms are based on real entities, nor that they are coherents. This is a pluralist conceptions since, for the implicationist, anti-pluralism is a valid stance only for logic. From this point of view, (first order) logic is the background theory in which we study all the possible axiomatic systems. In fact, instead of studying mathematical proposition like φ, where φ is a theorem, the implicationist, that believes that these propositions do not have fixed truth values, studies propositions of the form $A \rightarrow \varphi$, where A is a finite conjuction of axioms that logically implies φ. This position is defended mainly ny Putnam and Russell, and the flaws that make it inadequate are known. Hamkins refines this position taking into consideration only coherents axioms, and from this point engage skolemism. The skolemism is based on the fact that every consistent theory has a countable model and all these models are on the same truth level from a mathematical point of view (that is, we cannot consider a model "truer"). From this, a skolemist can prove that there are some mathematical propositions that don't have a fixed truth value. For example Skolem, as we already seen in the previous section, arrived to the conclusion that the propositions on the countability don't have fixe truth value, while von Neumann denied any determinacy also for the propositions about finitness. The independence results had strenghtened this position. Hamkins doesn't formulate this position from a proof theory point of view (e.g. using Hilbert system), but develops it using a model theory: the conception of the broad multiverse.

In the next sections I will discuss in some detail Hamkins' position. After some historical remarks about the multiverse (section 1.1.1), section 1.1.2 will be dedicated to the main method to produce extension from a given axiomatic system. After that I will discuss a parallel between set theory and geometry (section 1.1.3). Aferwards, in section 4.2 there will be some remarks about categoricity, the main argument against the multiverse. Before concluding with some general consideration on the Hamkins' position (section 1.1.6), in section 1.1.5 I will try to formulated (following Hamkins' reasoning) axioms to formalize the broad multiverse.

Before proceding, we need to precisate the "ontological" nature of these universes. At the start of this chapter I have said that, usually, the multiverse conception is an anti-platonic position. In fact, we can consider those who maintain this position not antiplatonic *absolutist*, but *relatively* antiplatonic when compared to those that maintain the position of the Single Universe (affirming the existence of a single mathematical universe is a form of Platonism definitely stronger than affirming the existence of various alternative universes). In fact, and this is true in a particular way for Hamkins, even for those who follow the broad multiverse the individual universes *exist*, from a platonic point of view. Hamkins' position is a higher order realism (as Platonism applied to universes), that is a realism that affirms the actual existence of these alternative universes, that our instruments of set theory allow us to explore. This means that you can not reduce the concept of the multiverse to more imaginative formalism: we always prefer some universes (although not at a theoretical level, but only at a practical one), and we are not obliged to consider them all at the same level. This makes it possible to avoid the problems mentioned in the previous section: the universes that are incompatible with our background theory can be considered "worse" than those compatible, without denying theirs existence within the multiverse. The property that we were looking for in the previous section become the incompatibility: all universes, both compatible and incompatible, exist, but only those that are compatible are considered, while those incompatible dropped out (since incompatible). Finally, among compatible universes, we can make a further choice, according to the research objectives that guide us (for example, if we are studying certain large cardinals we could prefer certain universes).

Wanting to clarify even more Hamkins' platonism, we can say that it is a *fullbloaded* Platonism. According to this position, mainly developed by Balaguer, *any* set theory describes a domain of actually existing objects. The notion of truth does not vary from that usually used in mathematics. Of course, it implies that the consistency of a mathematical proposition is enough to decide on its truth. In common practice, to state that a statement is true is to say that it is true in the *real* universe of sets. But we have seen that in accordance with the position described there is more than one universe. This however does not change our conception of truth: a proposition is true if and only if it is true in the real universe. But this means that because there are so many concepts of sets, each consistent theory will be true for some real universe. For example, according to this position, both $ZF + \neg C$ and ZFC describe true parts of mathematics. The fact that they are in contradiction is not a problem: in fact, they describe *different* parts mathematics. We could then say that ZFC describes the Earth-1 universe, while $ZF + \neg C$ the Earth-2 universe. In the Hamkins vision, each individual universe instantiates a single concept of set, built from a number of axioms. In our example, the Earth-1 refers to the universe (or region of the multiverse) that instantiates the concept of set expressed by the axiom of choice, while Earth-2 instantiates the concept of set that denies the Axiom of Choice.

1.1.1 Precursos: von Neumann and Mostowski

The first hints to the conception of the multiverse are found in an article by von Neumann in 1925, " Eine der Axiomatisierung Mengenlehre ". In this article von Neumann takes into account the situation in which a model of set theory could be a set within another model. Moreover he points out that this set could be "finite" in the first case and infinite in the second. Similarly, an ordering in the first case may be well founded, while in the second case could be ill-founded. He concludes that these facts weaken the position of set theory and that is difficult to find a way to redeem it. The article by von Neumann would seem to suggest precisely the position of the multiverse: more than a fact suggests to von Neumann that set theory cannot be considered, monolithically, a candidate for the foundation of mathematics, and that new facts will hardly reverse the trend.

In 1967, in the aftermath of independence results from Cohen, Kalmár said "I think in the future we will say *lets take a set theory* with the same simplicity as we now say lets take a group G or field F". The same year Mostowski, during a conference, stated that there are "many essentially different notions of set that are equally eligible as intuitive bases for set theory".[4]

Now, as these hints may seem not relevant enough, we must consider the historical context of the evolution of set theory. Until the '20s mathematicians had lived in the illusion that the fundamental propositions of set theory would be gradually decided within the theory. If the paradoxes arising from the choice could have disturbed the consciences of many mathematicians, and the incompleteness theorems had shattered the dream of a foundation of mathematics, it was with Cohen's independence results we realized that the universe described by set theory it was not necessarily one and monolithic. Of course, if the independence results had been limited to metalogic proposictions, and few real important propositions had been decided with the addition of certain self evident principles, then the story would have been different, and probably the mathematical universe as a Single Universe would not be questioned. But these are just guesses, the story had followed a very different path. With the development of the most powerful techniques, such as forcing, inner models and ultraproducts, we have discovered alternative universes, but at the same time perfectly legitimate.

Of course, this is not enough to make obsolete the concept of a single universe. We have already seen how the multiverse is problematic, and in chapter 4 we will see how the position of a single universe is solid, and its arguments valid. Indeed, the central fact that derives from all of this is that it is not possible to solve this problem from a purely mathematical point of view. Now the alternative universes "exist", have been explored and visited, deeply developed. There cannot be a single mathematical result that would eliminate all of them with a sponge. Similarly, there can not be a mathematical result that removes definitively strength to the arguments for a single universe.

[4]This can be found in [Lakatos 1967].

1.1.2 Forcing Ontology

So far, talking about the multiverse, we were at a very abstract level. Not only we haven't described a single universe, but, more importantly, we have not described *how* to create these universes. As mentioned in the previous section, the techniques to create alternative universes are essentially three: forcing, ultraproducts and inner models. In this section we will cover the forcing, from an "ontological" point of view. This means that we shall question on the existence of extensions through forcing.

Intuitively, an extension of the universe V through forcing is a universe $V[G]$, which consists of the universe V with the addition of G. So the question we must ask is this: these extended universes actually exist or are illusory?

The situation may be similar to that of real numbers and complex numbers. We define more concretely, the extension by forcing as the extension through a filter V-generic G on the notion of forcing $\mathbb{P} \in V$. Obviously this filter *does not* belongs to V, but to $V[G]$. Therefore to state that there are no filters V-generic is perfectly legitimate (considering only V), just like to say that there are no square roots of -1. In the latter case, however, we are limiting our perspective to the real: if we look at the extension of $\mathbb{R}$, $\mathbb{C}$, we see that $\sqrt{-1}$ exists. So we must consider the extensions by forcing $V[G]$ exactly as we consider $\mathbb{C}$.

More precisely, to build an extension through forcing we first need a *base model* V of set theory (the advocates of the Single Universe consider *the* base model) and a partial ordering $\mathbb{P} \in V$. Supposing that $G \in \mathbb{P}$ contains elements from every dense subset of $\mathbb{P}$ in V (thus G is a V-generic filter), we build the extension $V[G]$ closed to the elementary operations of building sets:

$$V \subset V[G].$$

In other words, extending V through forcing means adding the element G to V, leaving everything elese unchanged. This is the exactly principle beyond field extensions like $\mathbb{Q}[\sqrt{2}]$. In fact, every object in $V[G]$ has a name in V, is built from its name and from G. From this we can prove that $V[G]$ is always a model of ZFC, although it can exhibit some truth that are different from the usual ones in set theory. This depends on the choice of $\mathbb{P}$.

Another way to proceed requires the definition of the *forcing relation* $p \Vdash \varphi$, that is satisfied when every V-generic filter G containing the condition p has $V[G] \models \varphi$. Summarizing, the following are the foundamental facts about forcing:

1. the extension through forcing $V[G]$ satisfies ZFC;

2. every proposition φ that is true in $V[G]$ is "forced" by some condition p in G;

3. the forcing relation $p \Vdash \varphi$ is definable in the base model (for a fixed φ or for a φ of fixed complexity).

From this hints on the behavior of forcing we can understand how important is the choice of the generi filter. We can use three methods to do it:

1. the method of the countable transitive base model;
2. the method of the boolean evaluated method;
3. the natural method.

Among these methods, the first one requires some work in the metatheory, while the other two allow us to stay in ZFC. Moreover the first method doesn't allow to force on any model of ZFC. Finally, the method more easily accepted by the advocates of the multiverse is the third one.

Using the first method, we start with a countable transitive model M and not with the whole V. That the model is transitive is a natural consequence if the membership relation is well-founded (by the Mostowski Collapse Theorem), while the countability of the model comes from the Löwenheim-Skolem Theorem. Since the model is countable, will have countable dense subsets of $\mathbb{P}$, that we can enumerate $D_0, D_1, D_2, \ldots$ and then take any condition $p_0 \in D_0$, then a condition (below p_0) $p_1 \in D_1$ and so on, so that we can build a descending sequence $p_0 \leqslant p_1 \leqslant p_2 \leqslant \ldots$, such that $p_n \in D_n$. From this it follows that the filter G generated by this sequence is M-generic. We can then build an extension $M[G]$ through forcing. This extension will satisfy $ZFC + \neg\varphi$, where φ is the proposition we want to prove to be not provable in ZFC. We can therefore say that $M[G]$ is a "atypical model" of set theory. But this method has some problems: first, it is only applicable to some models of set theory (in particular, those uncountable). Secondly, we cannot prove the existence of these countable transitive models, since, for the incompleteness theorem, if ZFC is consistent then it cannot prove that there are models of ZFC (to do this youwe have to take into account the theory $ZFC + Con(ZFC)$, and so on ad infinitum). To avoid this problem you can consider, instead of the full ZFC, a fragment of it, ZFC^*. This solution, however, even if it avoids the requirement of existence of countable transitive models, does not produce a $ZFC + \varphi$ model, but only proves that this theory is consistent (the models will be produced using another technique, for example through Henkin's construction). Because of these reasons this method of forcing is not really useful to the cause of the multiverse.

The second method, the boolean evaluated model, is not exclusive of set theory. In fact, it can be used for any first-order theory, applying equally to groups, rings, graphs, etc. Being applicable to any first-order theory makes it possible to develop a forcing to set theory without having to introduce generic filters and dense sets. A boolean evaluated structure consists of a complete Boolean algebra $\mathbb{B}$ with a collection of objects, called *names*, and the assignment to the atomic formulas $\sigma = \tau$ and $R(\sigma, \tau)$ of elements of $\mathbb{B}$. This assignment is the Boolean value of the objects (names), and is denoted as $[\![\sigma = \tau]\!]$ and $[\![R(\sigma, \tau)]\!]$. Also this assignment has as purely technical requirement that the axioms of equality are respected. We can then extend the boolean values to all formulas inductively. So, concretely, to specify a boolean evaluated model we need a set of objects and the definition of the Boolean value of the atomic relations on them. The rest follows recursively. As for set theory, you start with a universe V and a complete Boolean algebra $\mathbb{B}$ in V. At this point we define inductively the set of names, so that τ is a $\mathbb{B}$-name if and only if it consists of pairs σ, b, where σ it is a $\mathbb{B}$-name previously constructed and $b \in \mathbb{B}$. The main idea of this method is that this name puts the set denoted by σ in the set denoted by τ with boolean value at least b. Continuing this process, we manage to build the boolean evalued structure $V^{\mathbb{B}}$. In this structure all of ZFC axioms have boolean value 1 and also, for a fixed formula φ, the function $\tau \mapsto [\![\varphi(\tau)]\!]$ is defined in V. Since this structure respects the deduction and contains no contradictions (in fact, the latter have as a Boolean value 0), and since the axioms of ZFC have boolean value 1, it follows that$ZFC + \varphi$ is consistent. Not even this method is totally suitable. It is better than the previous one, mainly because it is applicable to any ZFC model and it does not require the existence of special models unsatisfiable in ZFC (this avoids having to refer to increasingly strong theories). However, as with the previous method, neither this method produces a model for $ZFC + \varphi$, but merely shows his consistency.

The last method took into consideration, the natural one, is very similar to that of the boolean evaluated model. The similarity between the two methods, although not visible on the surface, it is due to the fact that the easiest way to prove the basic facts of the natural method is using boolean evaluated models (and in particular using the Boolean ultraproducts). As with the Boolean method, we start considering a particular universe V, and then affirm the existence of a V-generic filter G on $\mathbb{P}$. We Conclude the process "moving" in $V[G]$: all the assumptions and all that was true in V is retained, but the proof is carried out exclusively within the horizons of $V[G]$, without any reference to V. This method allows to consider initially an object within V and then argue in $V[G]$, forgetting that the object initially targeted belonged to V, but considering it exclusively within $V[G]$. Also it is retained all the knowledge we had about V, relativized to a predicate for V, but is adopted the perspective of the universe $V[G]$. So, although the actual existence of a V-generic filter is not proven, the effect is exactly that. From this point of view, this method is the one closest to the vision of the multiverse: even if the extension through forcing of V doesn't exist, we can act exactly as if it existed. In fact, whatever the universe V chosen, we can always use the forcing to change universe, moving to an extension $V[G]$.

Although *mathematically* there is no proof of the existence of these alternative universes, for the multiverse view these universes are very real. Sure, it's a position that you can never prove within the set theory, but as a philosophical position has its dignity. It allows us to interpret some of our experiences in the application of forcing, in which we work with objects that we cannot fully grasp. The situation is similar to that astrophysicist who postulated the existence of a planet not through direct observation, but noting the perturbations in the orbits of known planets. With forcing, we explain certain facts of known objects by postulating the existence of other objects "outside our field of vision".

1.1.3 The analogy between geometry and set theory

Surely the multiverse conception is peculiar: asserting the existence of alternative universes is always a debated position (e.g. the furious debate on the possible worlds or the multiverse in physics). But, from a mathematical point of view, is possible to draw a parellal with geometry.

As set theory, geometry was born by developing a unique concept of (physical) space. Later, mathematicians have proved theorems in what they believed to be a unique universe. As in the case of set theory, also in geometry results of independence have begun to crack this security: the search of the independence of the fifth postulate led to the development of non-Euclidean geometries. Initially considered just simulations, games, inside the " real 'geometry' ', only with time were finally considered perfectly legitimate.

Set theory has followed a development quite similar, although concentrated in a considerably shorter time. Like geometry, it was born developing a concept that was believed unique, the concept of set. At first, theorems have been proved within a single universe. The results of independence and the development of "generatiion techniques" for universes have pierced the veil, and instead showed the existence of alternative universes. Unlike geometry, these universes have not been accepted yet, and are still viewed with suspicion. If from the philosophical or mathematical point of view we have no grounds to say that this situation will change in the future, from the historical point of view, as we have just seen, we have more than one clue.

Other than the historical development, it is possibile to trace a parallel between set theory and geometry from the method of studying these alternative universes. In fact, in both disciplines, the approach to alternate universes follows three steps:

1. first, an alternative universe is built and studied as a simulation to a better understanding of a independent proposition in the "classical" universe;

2. then, the alternative universes starts to be considered per se, adopting some negation of the independent proposition;

3. finally, it becomes possible to study alternative universes from an abstract point of view.

1.1.4 Categoricity

At this point we have all the fundamental elements of the broader multiverse: we have a philosophical position (the existence of alternative mathematical universes), a mathematical method (which allows us to generate these alternative universes), and a similar situation in a different discipline (i.e. the presence of alternative universes in geometry). But we have to overcome an obstacle, categoricity. The importance of this argument does not derive solely from the fact that it is of a purely technical nature (it consists in fact of theorems), but above all by the fact that these theorems deny the very essence of the vast multiverse: essentially, these theorems state that all models possible are isomorphic, thus we are considering only one model.

The most important results on categoricity date back to the early days of set theory. This is the proof by Peano that his second order axioms (PA_2) characterize the unique structure of the natural numbers and the second order proof of Zermelo's set theory, which states that the possible universes are V_κ, where κ is an inaccessible cardinal. As you can see, both proofs deal with the second order, but the second one is more interesting (at least from our point of view). In fact, although the categorical nature of natural numbers is not unimportant (of course, the fact that there is only one concept of natural numbers may seem obvious, but as we will see even this is not so obvious), Zermolo's proof is undermining the roots of the multiverse.

In fact from Zermelo's proof we can derive the following argument, that supposily will prove that the universe of set theory is unique. Lets suppose that we can compare level by level through the use of ordinals two conceptions of set V and V'. At every level, if $V_\alpha = V'_\alpha$, then $V_{\alpha+1} = V'_{\alpha+1}$, and thus $V = V'$.

The problem with this argument is that supposes to be possible to compare two different concepts of set, but this is possible only considering an original concept of "membership", so a theoretical context in which the sets V and V' are compared. This means to assume a concept of membership, and thus of set, that is "general", in which then we can analyze the particular concepts of V and V'. But this radically transforms the conclusions of the proof: given a context, it is shown that, within a fixed meta-theoretical context, any universe, if it contains all the sets, is unique. But this is quite different from categoricity: this refers to a background context, and the comparison is in relation to this background. The categorical rather affirms the equality in an absolute sense.

Moreover, this argument assumes that the two universes agree on the concept of ordinal, but this is not as obvious as it might seem. In fact, it is enough to assume that the two universes agree on the concept of well ordering, but also this is far from obvious. These assumptions both draw strength from Peano's categorical proof. The classical counting operation$0, 1, 2, 3, \ldots$ is beyond any doubt. However, the Peano's proof is a second order proof, which cannot be considered outside the context of a fixed concept of subsets of $\mathbb{N}$. But this means that our ellipses, when we count, is charged with ontology borrowed from set theory: it is based on a certain concept of set, and is intended to include all sets of natural numbers, whose existence and nature supports and it is necessary for the proof itself.

The main point here is that the proof requires a context with a fixed concept of set, and so can take into consideration only a concept of absolute finite number. The fact that the finite numbers structure is uniquely determined in a given context depends on which subsets of natural numbers exist in that context.

Thus the categoricity arguments cannot undermine the multiverse conception and replace it with a Single Universe because, doing this, they have to suppose the existence of the Single Universe itself!

1.1.5 Multiverse axioms

So far our perspective was only philosophical: in fact we still don't have anything concrete on the multiverse, only philosophical reflections on why the existence of these alternative universes is possible. At best, we have a philosophical interpretation of a mathematical fact (forcing) that allows us to defend our philosophical positions on a slightly firmer terrain. But this is not enough. In this section I will try to formulate some formal principles for the multiverse.

Remember that, by the very nature of the multiverse, it will not be possible to formalize all of our principles in a first-order theory, and even in a second order language like the Bernays-Gödel theory or Kelley -Morse set theory. This is due to the fact that the very nature of the multiverse supposes the existence of alternative universes that can be totally different than the one in which our principles are formalized. So a formalization that makes sense in a specific universe could be totally insane in another. Luckly, some interesting facts can be formalized in the usual language of first order set theory, so what started as a simple philosophical reflection can also become a mathematical and rigorous enterprise.

First Hamkins does not want to place restrictions on the models to be included in the multiverse, either upwards or downwards. This way you can include both weaker theories as ZF or ZF^- (the Zermelo Fraenkel set theory without the extensionality), and much stronger theories such as $ZFC+$ large cardinals. So there is no restriction on what universes may exist, so that the multiverse is as large as possible. But, at any given time, we "live" in a single universe, and at any time we can move around in the multiverse. This freedom of movement does not mean that individual universes can access the entire multiverse. So, in any universe, there aren't principles of construction of sets that require the quantification on all the multiverse, unless this quantification can be reduced to be on sets within a single universe.

Since we want ou multiverse to be a inverted cone, that starts if a single universe at its base and from there starts expanding, the following is the most basilar principle:

Principle (of Existence)**.** *There exists at least one universe, and this universe is V.*

Obviously a single universe is not enough, so we have to introduce a second principle:

Principle (of Feasibility)**.** *For any universe V, if W is a model of set theory and is definable or interpretable in V, then W is a universe.*

This principle allows us to consider the built universes as actual existent universes.

The next principle will permits us to formalize the discussion about forcing of section 1.1.2:

Principle (of Forcing)**.** *For any universe V e any forcing notion $\mathbb{P} \in V$, there exists an extension through forcing $V[G]$, where $G \subset \mathbb{P}$ is a V-generic filter, and this extension is a universe.*

Untile now all the existent universes contain, like in the "classic" case, all the ordinals. But these ordinals are only *theirs* ordinals. So we want to permit the extension of single universes to higher universes, since we know that, given a universe, there are universes with much more ordinals:

Axiom 1 (of Reflection)**.** [5] *For every universe V, there exists a much higher universe W with an ordinal ϑ such that $V \precsim W_\vartheta \prec W$, where with $\precsim$ we denote the pre-ordering of W_ϑ on V.*

With the next principle we want to formalize the fact that there is no possibility of communication between two universes, in particular if they are located far away from a hierarchical point of view (i.e., in the case a universe is much larger than the other). The lack of communication and analysis of other universes is due to the fact that the background theory between two universes may vary, and therefore certain true notions in a given universe may be false in another. One of these notions is certainly the countability: in fact just with a change of context the same set could be seen both countable and uncountable. This comes from the Löwenheim-Skolem Theorem: there are countable models of set theory who are not aware, in fact, to be uncountable. We can then formulate this principle:

[5] I denote this principle "Reflection Axiom" because in set theory we can already find "Reflection Principles".

Principle (of Countability). *Every universe V is countable from the prospective of a larger universe W.*

Just like countability, also well-foundness is based on the chosen background theory. As every model thinks that it is the largest model possible, in the same way every model thinks that only its ordinals are well-founded. This is because in every single universe the theory in which the ordinals are well-founded vary, and all these theories cannot communicate. Thus we can formulate a principle the same way of the previous one:

Principle (of Well-Foundness). *No universe V, from the prospective of all the other universes, is well-founded.*

We now take into consideration the case of embeddings. Lets imagine that we are in the universe V and that we have the embedding $j : V \to M$, e.g. an embedding of ultraproducts. We can iterate further this embedding:

$$V \to M \to M_2 \to M_3 \to \dots$$

At the model M_n we will have the embedding $j : M_n \to M_{n+1}$ without knowing that has been already iterated infinite times. The following principle formalize this situation:

Principle (of Inverted Embeddings). *For every universe V and for every embedding $j : V \to M$ in V, there exists a universe W and an embedding h*

$$W \overset{h}{\longmapsto} V \overset{j}{\longmapsto} M$$

such that j is the iteration of h.

For the last principle lets consider the fact that every countable transitive model of set theory can be iterated in a model of $V = L$. In the multiverse, we don't want that every universe can be absorbed in L, so that every universe is a countable transitive model in a much larger L:

Principle (of L Absorption). *Every universe V is a countable transitive model of another universe W that satisfies $V = L$.*

At this point we have to question the coherence of this conception. At presente, we are able to prove the "limited" case: we can have a set of models of ZFC that satisfies all the closure properties of the principles described above. In fact the set of all saturated countable computable models satisfies all the principles proposed.[6]

[6]The proof is in [Gitman&Hamkins 2010].

1.1.6 Some arguments against Hamkins' multiverse

In this section I will discuss some arguments against Hamkins' multiverse.[7] There are essentially two arguments:

1. the first one attacks Hamkins' claim that the existence of different models can be interpreted in terms of existence of different universes of sets that instantiate different concepts of sets;

2. the other one argues against the characterization of models as alternative universes.

The first objection attacks the core of the concept of the multiverse: two models (universes) M and N are to be considered as two totally different worlds. For example, two extensions by forcing $V[G]$ and $V[H]$, where $G \neq$, build two different worlds. The problem is that in all extensions by forcing of a transitive model of the axioms of ZFC the truth value, from the arithmetic level, are left unchanged. So it's hard to imagine these two worlds so different, if the finite numbers are the same in the first and in the second. For example, our two universes might have two very different $\mathcal{P}(\omega)$, but basically the same set of 10 elements.

Hamkins denies the definiteness of the concept of natural number, but this is of little use against this objection. Even assuming any concept of natural number, the argument is still directed against another target. In fact, if the universes imagined by Hamkins share some (not all) sets, it becomes more difficult then consider them totally different universes, and not different characterizations of a single universe. If we assert that in every single universe are totally different from all other objects, then the determination by natural numbers fall, but the proponent of the multiverse has to prove *where is* the real difference between objects that, from every point of view, look the same.

The second objection stems from the fact that each model (universe) can be considered as *inside* of V. This, however, poses a difficulty not just to the multiverse: how these models can be in V if there is no "real" V? But above all, how can our knowledge of set theory's truths derive from alternative universes to V? The problem is that in order to affirm the existence of a domain populated by objects, we must have a deep understanding of how those objects are made and how is made that universe. The realist view of the multiverse is based solely on the fact that the consistency implies the existence. But if this is not true, then the very structure of the multiverse is no longer able to withstand.

This objection attacks the root of the multiverse: the existence of extensions by forcing is based only on the fact that the consistency guarantees the existence. Denying this fact, and the extensions will lose every ontological value and revert to being sets in V. Obviously this fact alone cannot make

[7] For a more detailed account see [Friedman 2015].

the position of the multiverse inconsistent: it is perfectly consistent. That extensions by forcing exist, is beyond any reasonable doubt. The problem is *imagining* that these extensions are *real* alternative universes, and this step does not seem justified.

Lets conclude with some general consideration on the "types" of mathematics. In fact, we can divide mathematics in two category:

1. *formal* mathematics, like group theory, ring theory or topology;

2. *concrete* mathematics, like arithmetic or analysis.

In regards to the first type, we define some very general properties that are then studied and applied in various structures. For example, you can consider the commutative property in group theory, so defining Abelian groups. These properties are interesting in themselves: we formulate axioms that abstract them and then we consider the axioms that *characterize* the class of structures we are interested in. To continue the example above, we formulate the groups' axioms and then we take into account the class of Abelian groups. Obviously asking about items such as *The* group or the *The* topological space or try to solve the problem of the axiom of commutativity does not have too much sense, since in formal mathematics we do not have a single structure, but a multiplicity of structures which differ in some key properties.

In concrete mathematics, the situation is just the opposite. The axioms are used to characterize a *fixed* structure, and not a class of structures. For example, PA_2 characterizes $\mathbb{N}$ up to isomorphism. In concrete mathematics we have the results of categoricity, completely absent in formal mathematics. Of course, these results are not enough to ensure the existence of a single structure. But the question is not so much the philosophical meaning of these theorems, but the fact that in these theorems in formal mathematical are absent. Also in concrete mathematics we have independent issues, i.e. propositions that are not provable from a given set of axioms. For example, $Con(PA)$ is independent from PA. This means that there are models in which the basic theory *and* the independent proposition are satisfied, and models in which the basic theory is satisfied but the independent clause is *not* (we have already mentioned the legitimacy of $PA + \neg Con(PA)$). But independence simply means that the given proposition is not provable from that axiom system, and says nothing about its actual truth value (often an independent proposition in a weaker theory is proved in an enhanced theory).

The situation of set theory and geometry is quite different. Geometry is born as a discipline of practical mathematics: Euclid's axioms described a fixed structure (the physical space) and one of the issues discussed was the independence of the fifth postulate. With the birth of non-Euclidean geometry, geometry is instead passed to formal mathematics: now the axioms are no longer considered characterizing *the* structure of geometry, but the hyperbolic or the Riemann geometry.

Set theory is in the same situation, except that it has not completed the transition from the concrete to the formal mathematics. Initially, it was believed that the axioms of set theory were describing *the* structure of mathematics. Now, there are various systems of axioms of set theory in competition: for example, ZFC and $ZF + AD$. These two systems of axioms describe very different structures: in the first one is possible to prove the Well Ordering Theorem, in the second we cannot have the Generalized Continuum Hypothesis. A possible evolution of set theory might just lead to a situation comparable to that of geometry: a certain system of axioms will no longer have a higher status than the others and we will study the individual axioms systems as if they were non-Euclidean geometries and Euclidean geometry.

1.2 The anti-realist multiverse

In this section I will discuss the anti-realistic vision of the multiverse. According to this view, the individual universes do not exist ontologically, thus all the problems related to the effective existence of alternative universes are no longer interesting. In fact, according to this position, the multiverse is not a structured and independent reality, but only a phenomenon that arises in the practice of set theory. Consequently, there is no interence in talking about a "real V" or about the "multiverse": these are in fact only labels that can be used if necessary. In general, this position is especially useful in producing independence demonstrations since the multiverse is primarily seen as a tool to do it.

In the following sections, I will go into the details of the most important conceptions of the anti-realistic multiverse. In the section **??** I will discuss Carnap's position. This position is probably the closest one to the realist multiverse: for Carnap, every theory is worthy of being chosen to describe mathematical universes, and to prefer one theory to another is just a matter of *convenience*. A more moderate position is Shelah's one (section **??**), that is based on different degrees of "typicality" of the axioms' models. Concludes the chapter a discussion on Feferman position (section **??**), that we can consider the "standard pluralism". According to Feferman, since the concept of set is essentially *vague*, as well as that of the "linear" continuum, therefore the construction of the multiverse is inevitable.

1.2.1 Carnap's pluralism

Carnap's position[8], as referred (initially) only to logic (and, in the widest possible sense, mathematics and philosophy), it can also be applied to our case. As already mentioned, in Carnap vision any theory is legitimate, and adopting one over another is just a matter of convenience. For example, there is no difference between $ZFC + CH$ and $ZFC + \neg CH$, and you may prefer the one over the other indiscriminately. This position is thus essentially formalist: given that the meaning of the axioms is not dependent on any prior knowledge, the only theoretical motivation to prefer a certain group of axioms over another is the curiosity to see what it can be proved by those axioms. The main difference with the Hamkins' position (which also stated that there are no theoretical reasons for preferring one theory over another) it is how it's considered the notion of truth within their system: while for Carnap there is no "external" truth to the theory considered, for Hamkins is still possible to consider certain propositions *always* true or false, in any theory considered. In other words, whereas in Carnap every theory is independent from the others, in Hamkins all theories have a common core of true propositions, and differ only in the independent issues, such as CH.

Carnap's proposal is divided in three main points:

1. logic and mathematics are analythic disciplines and thus are without content, purely formal;

2. the meaning of the foundamental terms is determined only by the postulates that rule them, thus any set of postulates is equally legitimate;

3. the goal of philosophy is the study of the syntax of the scientific language's logic.

The claim that is the most interesting for our discussion is the second one.

First, it must be said that Carnap's pluralism is very radical. Of course, not as radical as the realist multiverse position (to state the actual ontological existence of alternative universes remains the most radical position possible), but the most radical of all the anti-realistic positions that we will discuss. This is because, for Carnap, there is no difference between a theory and the other, if not the "measured" one from the theorems that you are able to prove in it. For example, for Carnap, between $ZF + \neg C$ and ZFC there is no theoretical difference, but exclusively a "quantitive" one: in the second it is in fact possible to prove many more results than in the first.

Carnap's argument in favor of this version of the radical pluralism is based on the analyticity of mathematics and its lack of content. This is because, if the mathematical truths are without content and if it is possible

[8] Carnap's considerations can be found in [Carnap 1935], while an extensive commentary in [Koellner 2009].

to maintain this hypothesis for any arbitrary formal system, then it can be said that even apparently incompatible systems are actually perfectly compatible (because there is no conflict of content). But Carnap fails to prove that mathematical truths are without content, and this leaves us only with techniques insufficient for the purpose. The system developed by Carnap fact lacks a metalanguage able to demonstrate that two systems S_1 and S_2 seemingly incompatible actually have a void content and therefore are compatible.

In conclusion, the problem of Carnap's position is that it totally ignores the most fundamental results in logic and set theory (remember that his work appeared in 1935, and had not yet incorporated the latest results of Tarski) and then his position is disconnected from the actual developments of mathematics. In fact, is a very similar formalization to the system of Wittgenstein's *Tractatus*, but this makes it a formalization based solely on philosophical considerations (the main hypothesis, namely that mathematics is without content, is unproven), and without any demonstrative power (the main result that should be proven with this system, which is that two incompatible theories are actually compatible, is beyond the reach of this system). So we can say that Carnap's position is reduced to general criteria for adopting one theory over another.

1.2.2 Shelah's dreams: the mild formalism

As we have seen, Carnap's position is pretty sterile in terms of mathematics. Among mathematicians, one that comes closest to Carnap's position is Saharon Shelah. Its position[9] can be described as a *mild* formalism. Formalism, as it does not believe in the existence of a mathematical universe that we "discover" as we prove theorems. Mild, as it does not believe that mathematics is reduced to the simple manipulation of symbols and furthermore, does not believe that all consistent theories are at the same level.

Such a position is similar to that advocated by Hamkins (also for him the theories are not all equal, but some are preferable to others), with the difference that in Shelah the resulting models are not actually existing alternative universes. Shelah take into consideration the models of the axioms (in our case, the models of ZFC) depending on their "degree of typicality": some will be more "typical" than others because they satisfies less strong propositions, and thus more easily met by a greater number of models. For example, a model that satisfies $\exists\alpha[2^{\aleph_\alpha} = \aleph_{\alpha+\alpha+7}]$ is more typical than one that satisfies $\forall\alpha[2^{\aleph_\alpha} = \aleph_{\alpha+\alpha+7}]$, which instead will be atypical. We can also define certain propositions as typical and atypical: the first will be those usually satisfied by all models, the second is the ones never satisfied. Obviously, some propositions cannot be classified in this way: for example,

[9] It is developed in two articles, [Shelah 2002] and [Shelah 2003].

we do not know if the CH is typical or atypical (in a nutshell, is just another way of saying that is independent). In general, Shelah believes in what we might call a "theory of zermeloids". In other words, Shelah considers set theory and algebra very similar: as in the latter the object of study can be the groups, from which we derive a theory of groups, so in set theory the objects under study are the sets that satisfies ZFC, and so we have a "theory of zermeloids". We can also imagine alternative set theories, simply by changing the object of study we are interested in: for example, the study of sets that satisfy $ZFC + \neg CH$ could be called "theory of woodinoids".

Field [10] è della stessa opinione: possiamo sempre preferire che certe proposizioni siano soddisfatte dal nostro modello invece che non esserlo, ma non possiamo poi usare questa preferenza come prova del fatto che le proposizioni in questione hanno proprio quel valore di verità. Ad esempio, possiamo preferire dei modelli in cui CH sia soddisfatta, ma questo non significa che possiamo usare questa preferenza per argomentare a favore della CH. Anche Balaguer [11] è vicino a questa posizione: per lui infatti molte dispute matematiche non sono altro che dispute sulla soddisfacibilità di una certa proposizione all'interno di un modello standard, oppure (come nel caso della CH), dispute tra fautori di $ZFC + \varphi$ e fautori di $ZFC + \neg\varphi$.

Field[12] is of the same opinion: we can prefer that certain propositions are satisfied by our model, but we cannot then use this preference as evidence that the statements in question havethat actual truth value. For example, we can prefer the model where CH is satisfied, but this does not mean that we can use this preference to argue in favor of the CH. Also Balaguer[13] is close to this position: for him many mathematical arguments are nothing but disputes on satisfiability of a certain proposition in a standard model, or (as in the case of the CH), disputes between proponents of $ZFC + \varphi$ and proponents of $ZFC + \neg\varphi$.

1.2.3 Feferman's standard pluralism

The last of the anti-realist positions on the multiverse is Feferman's[14] According to Feferman, since the definition of *set* is essentially vague, it appears that even the "linear" continuum is vague. Consequently, in the daily practice of mathematics, we are forced to accept the construction of the multiverse, especially because we could never reduce the truth in the multiverse to the truth in a single defined universe. Any attempt to solve an independent proposition is impossible.

[10] In [Field 1998].

[11] In [Balaguer 1995] e [Balaguer 1998].

[12] In [Field 1998].

[13] In [Balaguer 1995] and [1998 Balaguer].

[14] In a series of articles: [Feferman 1999], [Feferman 2000] and [Feferman 2011].

For Feferman, this vagueness is inherent to the concepts taken into account: there is no reasonable way of specify the notions of set (and thus the notion of continuum) examined without greatly changing the objects that these notions want to define. For example, consider all the arbitrary subsets of real numbers. Assuming that these subsets are all in L or even in $L(R)$ may be one of these details we want to specify, but would violate our definition of "arbitrary sets".

But we must note that Feferman's pluralism only applies to the truths about the levels of the cumulative hierarchy *above* V_ω, while all *finite* mathematics (thus all the truths concerning cumulative hierarchy below V_ω) has a precise and anything but vague meaning (for example, for Feferman, the concept of natural number is not vague, while the concepts of arbitrary set of reals it is).

In conclusion, we can state that Feferman's pluralism is a constructivism that, because of its nature, is in difficulty with the higher levels of the cumulative hierarchy.

Capitolo 2

The Generic Multiverse

The pluralist position described in the last chapter is too extreme to be a foundation for set theory. As already explained, the main flaw of the broad multiverse is that is too big for its porpuse. The only way to procede in our research is to consider more moderate positions. To do that, we have to limitate the class of the universes using strong logics (e.g. the β-logic or the ω-logic). For example, we can restrict the multiverse to just the universes that are β-models or ω-models. One consequence of choosing the ω-models as universes is that the proposition $Con(ZFC)$ will be true (although this could change with the background theory chosen), while the fact that all the projective sets are Lebesgue measurable will be undetermined (if we stick with the broad multiverse, both the propositions would be undetermined). The main reason the develop this position, called *generic multiverse*, is the will to consider determined the large cardinals axioms and the definable determinacy, while propositions like the CH will stay undetermined.

The notion of truth in the generic multiverse is very similar to the one in the broad multiverse: a proposition φ is true if and only if is true in *every* universe, it's false if and only if is false in every universe and in any other case (in some universe is false, in other universes is true) is nudetermined. We can formalized this definition in V: for every φ there exists a proposition φ^* such that φ is true in every universe of the generic multiverse if and only if φ^* is true in V. This last definition is the difference between the notion of truth in the generic multiverse and the notion of truth in the broad multiverse (we cannot consider this definition in the borao multiverse because there is no fixed V): without it the generic multiverse would be a broad multiverse, only a little bit smaller. While in the broad multiverse there is no restriction to the universes admitted in the multiverse, every universe is automatically admitted, in the generic multiverse there is a very tight choice.

Lets consider this position with a little more details. Lets take as background theory the theory $ZFC+$ "There exists a proper class of Woodin cardinals". This assumption assures the truth of the axiom for the definable determinacy (for example $AD^{L(\mathbb{R})}$ or PD), so this theory is suitable to investigate unvierses where large cardinals axioms and definable determinacy have fixed truth value. Lets defined the generic multiverse $\mathbb{V}$ as the closure of V under enlargement and inner models. Given a countable transitive model M, the generic multiverse based on M is the smaller set $\mathbb{V}_M$ such that $M \in \mathbb{V}_M$ and for every couple of countable transitive models $\langle N, N[G]\rangle$, such that:

- $N \models ZFC+$ "There exists a proper class of Woodin cardinals";
- $G \subset \mathbb{P}$ is N-generic for a any partial order $\mathbb{P} \in N$;

if one among N and $N[G]$ is in $\mathbb{V}_M$, then they are both in it.

In the next sections I will describe the two main positions of the generic multiverse: Woodin's position (section 2.1) and Steel's position (section 2.2). Although they are both very similar, there is an important difference:

- for Woodin the multiverse is plausible, but improbable;
- for Steel instead the truth of the multiverse conception is evident, and also is possible to redux the truth in the generic multiverse to the truth in a fragment of it (Steel class this fragment the *core* of the multiverse).

From a superficial point of view both Woodin and Steel would seem advocates of the Single Universe conceptions. But, since thei thought on this point is not always clear (they haven't expressed any preferences between the two positions) and their contributions are always about the multiverse, choosing a label for them will always feel arbitrary. For these reasons I will consider them under the most obvious one, that is the multiverse one.

2.1 Woodin's position

Let's start with some very basic definitions: with *multiverse*, we refer to the collection of possible universes of sets, and a proposition is true if and only if it is true in each of these universes. So far, nothing new compared to the broad multiverse. We define the multiverse *generic* if and only if it is generated from each universe of the collection by closure under enlargement and inner models. Like with the broad multiverse, here we are using the Cohen forcing to generate our multiverse: in fact, as we have seen, this is the fundamental technique to build non-trivial extensions of a given countable model of ZFC. Unlike in the case of the broad multiverse, here we only consider these universes thus generated (in the broad multiverse instead we were accepting all possible universes). For example, suppose that M is a transitive countable set such that $M \models ZFC$. Lets $\mathbb{V}_M$ be the smallest among the countable transitive sets such that

1. $M \in \mathbb{V}_M$;

2. every couple $\langle M_1, M_2 \rangle$ of countable transitive set are formed as such:
   item $M_1 \models ZFC$;M_2 is a generic extension of M_1.

If $M_1 \in \mathbb{V}_M$ or $M_2 \in \mathbb{V}_M$, then both M_1 and M_2 belong to $\mathbb{V}_M$. Moreover, if we define $\mathbb{V}_N$ in the same way as $\mathbb{V}_M$, but with another set $N \neq M$ instead of M, then we can easily prove that $\mathbb{V}_N = \mathbb{V}_M$. The set $\mathbb{V}_M$ now defined is the generic multiverse in V generated by M.

2.1.1 The truth in the generic multiverse

As we have seen in the case of the multiverse, the notion of truth by which a proposition is true if and only if it is true in every universe of the multiverse has a big problem: in fact, requires a larger universe in which compute the truth and the multiverse itself (in other words, it needs a universe in which the truth is defined and then the same truth will be applied to the multiverse). This issue (and the infinite regress that results) does not exist in the case of generic multiverse. In fact, for every φ there is a φ^*, depending recursively from φ, such that φ is true in every universe of the generic multiverse generated by V if and only if φ^* is true in V. Mind that the transformation that sends φ in φ^* is explicit, so you have to specify it in each case, and also *does not* depend on V. Returning then to the case before, with M and $\mathbb{V}_M$, the following statements are equivalent:

- $M \models \varphi^*$;
- $N \models \varphi$ per ogni $N \in \mathbb{V}_M$;
- $N \models \varphi^*$ per ogni $N \in V_M$.

This means that the concept of truth is not dependent on the metauniverse that defines the generic multiverse. An important consequence of this fact is that the definition of truth is the same in each universe of the generic multiverse, so we can investigate the truth of a certain proposition from *any* universe within the multiverse generic (instead in the case of the broad multiverse this was not possible, since the notion of truth varied from universe to universe).

But we need to better define what are the true propositions throughout the multiverse. In fact, the truth defined above is the truth *inside* a single universe, but now we must rise to the metatheoric level and define when a meta-theoretical assertion *on* the multiverse is true. Lets φ be a Π_2-proposition of the form "For every infinite ordinal $\alpha, V_\alpha \models \varphi$" for some proposition φ. A Π_2-proposition is a *truth of the multiverse* if and only if it is true in every universe of the multiverse itself, while we have to resort to large cardinals to define the "height" of the universe considered (in other words, if

$\alpha < \beta$, then V_{beta} will be "higher" than V_α). Note that this kind of notion of truth only applies to the Π_2-propositions, and not to the Σ_2-propositions (i.e. propositions expressible as a negation of a Π_2-proposition), since the definition of the generic multiverse is restricted to those universes generated by forcing on the sets (Cohen forcing), without taking into account the forcing on classes (Easton forcing). The problem of the Easton forcing is that it cannot be extended to the class of all cardinals, and therefore does not preserve the existence of large cardinals throughout the multiverse. Since the large cardinals occupy an important place within the generic multiverse, it is impossible to give up them, and then we prefer to avoid using Easton forcing (and thus also Σ_2-propositions).

Now we have to define the truth of a sentence in a given universe. Until now, we have state the most general definitions, without considering the conditions of truth within a single universe. Suppose a universe in which there is a Woodin cardinal[1], and let's say δ_0 is the smallest Woodin cardinal and that there is a proper class of Woodin cardinals (in this way the existence of δ_0 is constant throughout the multiverse). We denote by $H(\delta_0^+)$ the set of all sets X whose transitive closure has cardinality at most δ_0. The truths of the multiverse about $H(\delta_0^+)$ are those proposition φ true in the set $H(\delta_0^+)$ of each universe. Note that for every sentence φ saying

$$H(\delta_0^+) \models \varphi,$$

is a Π_2-proposition. The same holds for $H(\delta_0^+) \not\models \varphi$. So in each universe of the multiverse the set of all proposition φ such that $H(\delta_0^+) \models \varphi$ is the *theory* of $H(\delta_0^+)$, according to the computation of that particular universe. In other words, a set of all statements that satisfies a *fragment* of a universe is a truth of the multiverse if and only if it satisfies the same fragment in all individual universes. In addition, this set will be recursively within the set of true Π_2-propositions in that universe, while the set of φ such that $H(\delta_0^+) \not\models \varphi$ will not (for the Tarski Theorem).

We can now level up and no longer consider the truth in a fragment of a single universe, but the truth within a fragment of the whole multiverse (with fragment of the multiverse we mean a subset of its universes, for example $V_{\omega+\omega}$). Suppose the same things as in the previous case, that are the existence of a Woodin cardinal and the existence of δ_0. The statement "δ is a Woodin cardinal" will be equivalent to the statement $V_{\delta+1} \models$ "δ is a Woodin cardinal", so $\delta = \delta_0$ iff $V_{\delta+1} \models$ "$\delta = \delta_0''$. Consequently, assuming the existence of a Woodin cardinal, for every proposition φ the statement $V_{\delta_0+1} \models \varphi$ is a Π_2-proposition, as well as the proposition $V_{\delta_0+} \not\models \varphi$. So,

[1]That is a cardinal such that for each function $f : \delta_0 \mapsto \delta_0$ there exists a cardinal $\kappa < \delta_0$ with $\{ f(\beta) \mid < \kappa \} \subseteq \kappa$ and an elementary embedding $j : V \mapsto M$ from the von Neumann universe to a transitive inner model M with critical point κ (i.e. κ is the smallest cardinal who does not map itself) and $V_{j(f)(\kappa)} \subseteq M$.

in every universe of the multiverse, the set of all sentences φ such that $\mathbb{V}_{\delta_0+1} \models \varphi$ is the *theory* of V_{δ_0+1} in the computed form in that given universe. In other words will be the metatheory in the perspective of that universe. Like in the case of the fragment of a universe, this set will be recursive in the set of true Π_2-propositions, while the set of φ such that $\mathbb{V}_{\delta_0+1} \not\models \varphi$ will not (for the Tarski Theorem).

In conclusion, according to Woodin is clear that in the generic multiverse, both at the meta-theoretical level and at the theoretical one, the truth of the whole always comes down to the truth of a fragment. This conclusion, however, is neither what a se theorist expects nor what wants. In fact, such a view reduces the whole truth of the transfinite universe to the truth in a fragment of that universe. But this for set theory is too simply, because you end losing many details available only to the highest levels of the transfinite. In fact, not only this is an understatement, but Woodin proves also its falsity.

2.1.2 The Ω Conjecture and the generic multiverse

According to Woodin, the notion of truth just described is not optimal. The issue is correlated with the Ω-logic and, in particular, with the Ω Conjecture. The following theorem, proved by Woodin, correlates the truth in the generic multiverse with the Ω-logic;

Theorem 1. *Lets assume ZFC and the existence of a proper class of Woodin cardinals. Then, for every proposition* φ *that is* Π_2*, the following statement are equivalent:*

- φ *is a truth of the generic multiverse;*

- φ *is* Ω*-valid (a proposition* φ*, in the* Ω*-logic,* isvalid$ifandonlyifisalogicconsequenceoftheemptyset$).

The Ω-validity is also *invariant* with respect to forcing: this means that it remains the same in every extension of V (and thus, in the case of generic multiverse, remains the same in every possible universe). One consequence of this is that, given a background theory, the notion of the generic multiverse truth is "robust" compared to Π_2-propositions. In other words, the proposition "φ is indefinite in the generic multiverse" is *determined* in the generic multiverse, so you are not forced to rely, to determine the notion of truth, on different universes. In this way we avoid having to take the multiverse of multiverses, or other particularly complex conceptions. From this point of view the notion of generic multiverse truth is good, but problems arise when you take into account the Ω Conjecture.

In fact, the Ω Conjecture has some important consequences for the notion of truth in the generic multiverse. Lets

$$\mathcal{V}_\Omega = \{ \varphi \mid \varnothing \models_\varphi \varphi \}$$

and for every specifiable cardinal κ, lets

$$\mathcal{V}_\omega(H(\kappa^+)) = \left\{ \varphi \mid ZFC \models_\Omega \text{“}H(\kappa^+) \models \varphi\text{”} \right\},$$

where with $H(\kappa^+)$ we mean the collection of sets of hereditary cardinality less than κ^+. One of Woodin's main results is that, assuming ZFC and the existence of a proper class of Woodin cardinals, the set $\mathcal{V}_\Omega$ is Turing equivalent to the set of truth Π_2 of the generic multiverse. Moreover, the set $\mathcal{V}_\Omega(H(\kappa^+))$ is exactly the set of truth of the generic multiverse based on $H(\kappa^+)$.

The conclusion of the previous section, however, left us with undesirable behavior on the part of the generic multiverse's truth: the truth can be reduced to the truth of a multiverse's fragment. We have already mentioned that this is not a desirable outcome in set theory. The purpose of the following laws is precisely to avoid this reduction of truth, thus restoring the full power to all the transfinite.

Principle (Truth Constraint)**.** *For any cardinal, the* Π_2*-truths of the generic multiverse* are not *recursive in the set of truth of* $H(\delta_0^+)$ *(or, in the general case, of* V_{δ_0+1}*).*

This law, by preventing the reduction of the truth of the multiverse to the truth of a fragment, captures the idea that the multiverse is so rich and complex that it can not be described from below. In other words, it prevents the truth (or even just the Π_2-truth) to be described in a specific fragment of the multiverse. In fact, if the set of Π_2-truths of the multiverse is recursive into the set of truths of the multiverse regarding $H(\delta_0^+)$, then with regard to the evaluation of the Π_2-propositions the multiverse is equivalent to the multiverse reduced to the fragments $H(\delta_0^+)$ of the universes of the multiverse. This does not mean that the truth will vary from universe to universe (as happened in the broad multiverse): the notion of truth is always the same, in all generic multiverse. It is simply not possible, within a single universe, defining the truth of the entire multiverse (but, metatheoretically, it is possible to notice that the notion of truth is the same in each universe). Also note that, for the Tarski Theorem, this law is trivially satisfied by the notion of "standard" truth of set theory (that is, the conception of the multiverse containing only one item, V).

Before we can state the next constraint, we have to give a prelimanry definition. A set $Y \subset V_\omega$ is *definable* in $H(\delta_0^+)$ (or in V_{δ_0+1}) in the whole multiverse if it is definable in the structure $H(\delta_0^+)$ (or in the structure V_{δ_0+1}) of every universe of the multiverse.

Principle (Definibility Constraint)**.** *For every specifiable cardinal* κ*, the* Π_2*-truths aren't* definable *in* $H(\delta_0^+)$ *(or in* V_{δ_0+1}*) in the whole multiverse.*

Also this law is to prevent the reduction of truth to the truth of a fragment of the multiverse. In fact, from the definition of definability as given above, it derives the ability to define any set from below. Moreover, as in the previous case, this law is, simply, satisfied for V. In fact, if the only universe in the multiverse is V, the set of truths of the multiverse about $H(\delta_0^+)$ (or V_{δ_0+1}) is simply the set of all true statements in $H(\delta_0^+)$ (or in V_{δ_0+1}) and the set of the Π_2-propositions which are truths of the multiverse is simply the set of Π_2-propositions true for V.

The situation of the notion of truth has now changed slightly. Although it remains unique throughout the multiverse, it is no longer reducible to the truth of a fragment. So now, in order to investigate whether a given proposition is a truth of the multiverse or not is no longer enough to control a fragment of it, instead we must turn to the whole multiverse.

The problem arises when we try to correlate the Ω Conjecture with these laws. The main results of Woodin show that the notion of truth of the generic multiverse *does not* respects neither the constraint of truth nor the constraint of definability.

Since the set $\mathcal{V}_\Omega(H(\delta_0^+))$ is exactly the set of the truths of the multiverse about $H(\delta_0^+)$, the requisite to satisfy the first law of the multiverse (the Truth Constraint) is, for the notion of truth of the generic multiverse, $\mathcal{V}_\Omega$ is not recursive in the set $\mathcal{V}_\Omega$. But the following theorem disprove this.

Theorem 2. *Lets assume the existence of a proper class of Woodin cardinals and that the Ω Conjecture is true. Then the set $\mathcal{V}_\Omega$ is recursive in the set $\mathcal{V}_\Omega(H(\delta_0^+))$.*

So, assuming the existence of a proper class of Woodin cardinals *and* that the Ω Conjecture is true the notion of truth of the generic multiverse violates the Truth Constraint.

There is a similar theorem about the Definibility Constraint:

Theorem 3. *Lets assume the existence of a proper class of Woodin cardinals and that the Ω Conjecture is true. Then the set $\mathcal{V}_\Omega$ is definable in the set $\mathcal{V}_\Omega(H(\delta_0^+))$.*

Consequently, if there is a proper class of Woodin cardinals and the Ω Conjecture is true, then the generic multiverse claim (that only the Π_2-propositions that are true are true in every universe of the multiverse) violates the Constraint of Definability, since this set of propositions would be definable within $H(\delta_0^+)$ for the entire multiverse.

In other words, assuming the existence of a proper class of Woodin cardinals and that the Ω Conjecture is true in the multiverse generated by V leads to the violation of both the first law of the multiverse and the second. Then the generic multiverse would be equivalent to the reduced multiverse given by the structure $H(\delta_0^+)$ of individual universes that compose the multiverse itself. But this is the conclusion that we were trying to avoid by introducing the multiverse constraints!

2.1.3 Some solutions to save the generic multiverse

The results just described aren't that big ostacle for the advocates of the generic multiverse. Indeed, there are various way to save the generic multiverse:

- reducing the size of the generic multiverse;
- changing the laws of the generic multiverse, so the notion of truth doesn't violate them;
- denying the Ω Conjecture;
- changing the notion of truth of the generic multiverse.

We will see how all these solutions are flawed.

First, we try to reduce the size of the generic multiverse. The idea of this solution is that if the generic multiverse is so small to *overlap* with a fragment, the problem of the reduction of truth does not arise and the notion of truth of this generic multiverse would not violate the laws proposed. Consider then the generic multiverse $\mathbb{V}^*_M$, such that it is generated exactly as the "normal" generic multiverse is generated by M, but with $\mathbb{V}^*_M \subset \mathbb{V}_M$. In other words, we create a generic multiverse in the usual way, but making sure that it is properly contained in another generic multiverse. In this way, the new multiverse will be *smaller* than the original multiverse. The problem with this solution is that, in the interpretation of the truth, absolutely nothing changes: as a result, we will have the same problem of reducing truth to a fragment of the multiverse, and then we will have to propose again *the same* laws of the multiverse . All this leads to the following theorem:

Theorem 4. *Suppose tha M is a countable transitive set such that*

$$M \models ZFC + \text{"esiste una classe propria di cardinali di Woodin"}$$

and that

$$M \models \text{"Congettura } \Omega\text{"}.$$

Also, suppose that $\mathbb{V}^+_M$ is the multiverse generated from M and that is smaller than the standard generic multiverse. Finally, lets suppose that the same propositions that hold in M hold in this smaller multiverse. Then, the notion of truth of $\mathbb{V}^+_M$ violates the laws of the generic multiverse.

As you can see, we have the same problem of the classic generic multiverse, that is, given the same notion of truth of the classic generic multiverse, we will get the same reduction to a fragment of the same multiverse: if we have a smaller multiverse, the truth of that multiverse will be reduced to an even smaller fragment. In other words, the reduced multiverse may be equivalent to the fragment of the classic multiverse with regard to the size, but this does not mean that the truth will always be defined on that fragment: In this new case, the truth will be defined on a fragment proportionally smaller . There is a way out from this reduction: continuing to reduce the multiverse would lead to having a multiverse containing only V. But this multiverse would be in all respects equivalent to the single universe of classical set theory, so this solution does not need to defend the vision of generic multiverse.

But we can narrow the multiverse so that the laws of the multiverse can't violated. In this case, however, the restriction will regard the *size* of the multiverse (we have just seen that it does not work). Instead, we will reduce the generation method of the multiverse. In particular, by limiting the generic multiverse to be the multiverse generated only allowing homogeneous forcing (forcing with stronger closure properties), we will have a multiverse in which the notion of truth *does not* violate the laws of the multiverse. The problem with this solution (and, in general, with all the solutions that preach the restriction of the generic multiverse) is that you have to find a set of Π_2-propositions which, despite being declared true in the narrow multiverse, they are not in the generic multiverse. This is because, as always, we want to prevent the truth of the entire multiverse to be reducible to the truth of any part (in this case, however, the situation would improve slightly than before: we do not have a regression to the simple universe V, but a multiverse whose truth is definable in one of its fragment, and in this fragment the truth is independent, i.e. not definable in another fragment). The obstacle in developing this position is that, currently, there are no other propositions for the set of the Π_2-propositions than the true Π_2-propositions of V. So even in this case, the position of the multiverse is reduced to the classical universe.

Since is not possible to solve our problems changing the multiverse, what about changing the laws of the multiverse? For example, we can change the Definibility Constraint as such:

Principle (Strong Definibility Constraint)**.** *The set of the* Π_2*-proposition that are truths of the multiverse it is not* uniformly *definable in* $H(\delta_0^+)$ *for the whole multiverse.*

Then, the set of Π_2-proposition should not be definable not by *a single* formula nor by, like the previous law, by a set of formulas. Thus, the notion of truth of the multiverse *does not* violate this stronger law. But we still the problem of the first constraint.

That said, we can try to weaken the proposed laws of the multiverse: similarly to what we did with the reduction of the generic multiverse, we can try to reduce the scope of the constraints. This means reducing the fragment of the multiverse in a way that is too small to establish the truth of the entire multiverse. To do this, we replace $H(\delta_0^+)$ with $H(c^+)$, where $H(c^+)$ is the set of all sets X that have transitive closure of cardinality at most $c = 2^{\aleph_0}$. From this it is possible to prove that $c < \delta_0$, so $H(c^+) \subset H(\delta_0^+)$, then the fragment taken into consideration is a lot smaller than the original fragment. It is therefore possible to formulate the following laws:

Principle (Weak Truth Constraint)**.** *The set of* Π_2*-propositions that are truths of the generic multiverse* are not *recursive in the set of truths of* $H(c^+)$.

Principle (Weak Definibility Constraint)**.** *The set of* Π_2*-propositions that are truths of the multiverse is not definable in* $H(c^+)$ *for the whole multiverse.*

But, as we already said at the beginning of the chapter, also these weaker laws are violated by the notion of truth of the generic multiverse:

Theorem 5. *Assume that there exists a proper class of Woodin cardinals and that the* Ω *Conjecture is true. Then the set* $\mathcal{V}_\Omega$ *is recursive in the set* $\mathcal{V}_\Omega(H(c^+))$.

Theorem 6. *Assume that there exists a proper class of Woodin cardinals and that the* Ω *Conjecture and the* AD^+ *Conjecture are both true.*[2] *Then the set* $\mathcal{V}_\Omega$ *is definable in the set* $\mathcal{V}_\Omega(H(c^+))$.

So even narrowing the fragment under consideration, we have that the truth of the multiverse will always be reducible to that fragment. Consequently, this solution is not suitable.[3]

At this point it is clear how the problem cannot be solved by changing either the multiverse or its laws. A possible solution could be to remove strength to the Ω Conjecture, to attack the theorems directly (remember that these theorems are based on the assumption that the Ω Conjecture is true). For example, we can say that the Ω Conjecture is problematic precisely as the CH, and therefore should be considered *independent* just like the CH. But, as underlined by the following theorem, this is not possible:

Theorem 7. *Assume* $ZFC+$ *"there exists a proper class of Woodin cardinals". Then, for every complete boolean algebra* B,

$$V \models \textit{Congettura } \Omega \iff V^B \models \textit{Congettura } \Omega.$$

So, unlike in the case of Continuous Hypothesis, we cannot be prove, by forcing, that the Ω Conjecture is independent from $ZFC+$ "there is a proper class of Woodin cardinals". In fact, while the notion of truth of the generic multiverse considers the CH indetermined, does not consider the Ω Conjecture in the same way. So we cannot sustain this position, because we

[2]This second is necessary because, if not assumed, to prove the theorem we would have to restrict the property about the universally Baire sets. See [Woodin 2009] for the details.

[3]Even replacing the Weak Definability Constraint with a Weak Constraint of Strong Definability will leas to the same problem as before: the notion of truth would satisfy the Second Law of the multiverse, but *not* the first.

cannot save the multiverse stating that the Ω Conjecture is undetermined if the multiverse itself says otherwise.

But we can try to radicalize our position: the Ω Conjecture is determined, but it is *false*. But we cannot argue that either: the Ω Conjecture is invariant in the multiverse (i.e. its truth value is the same throughout the whole multiverse), so it is not reasonable to expect that it is determined and further that, if false, is refuted by some hypothesis about large cardinals (a large cardinals hypothesis is, for example, the following proposition: for some (any) cardinal α, $V_\alpha \models$ "there is a Woodin cardinal" holds). In fact, the metamathematical consequences of the Ω Conjecture derive from the fact that it is Ω-satisfiable in a non-trivial way. This means that there is a Σ_2-proposition that states that there is an ordinal α and a universe in the multiverse V^*_{alpha} such that

$$V^*_\alpha \models ZFC + \text{"there exists a proper class of Woodin cardinals"}$$

and that

$$V^*_\alpha \models \Omega \text{ Conjecture.}$$

Since this proposition is a Σ_2 one, assuming the existence of a proper class of Woodin cardinal implies that this proposition is invariant in the multiverse generated by V. So the defender of the generic multiverse cannot help but affirm the determination. But, as a result, it must also assert its falsity. But this would be difficult to defend: in fact, despite the discussion on the truth of the Ω Conjecture is still open, and it is still possible to argue that if the Ω Conjecture is false *then* must be refuted by some large cardinals hypothesis, currently the hypothesis that this Σ_2-proposition is true is more likely. The reason is that while there are many examples of propositions for which the absoluteness with respect to forcing is provable (i.e. you cannot change the truth value by forcing) and cannot be decided from the axioms of large cardinals, there aren't instead examples of Σ_2-propositions that have the same properties. Moreover, if the Ω Conjecture is true, then the existence of such propositions is *impossible*. Moreover, it seems very strange that there are large cardinals axioms to prove the non trivial Ω-satisfiability of the Ω Conjecture and that there are not large cardinals axioms than can prove that the Ω Conjecture is true. For all these reasons, this solution has to be abandoned.

You can also try to save the vision of the multiverse rejecting one of the two laws of the multiverse. The problem with this solution is that it makes inevitable the reduction of the truth across the multiverse to the truth of a fragment. If we decide to give up the constraint of truth, then the truth in the multiverse will be reduced, in the sense of Turing reducibility, the truth in $H(\delta_0^+)$, if instead we decide to give up the second law of the multiverse (the Definability Constraint) then the truth in the multiverse will be reducible, in the sense of definability, the truth in $H(\delta_0^+)$. Either way, it falls into the problem that made us formulate the two laws on the multiverse: the truth would be defined from below.

The last resort for the advocate of the generic multiverse is simply to accept the failure of the laws of the multiverse, and try to think about the truths of set theory that are beyond the Π_2-truths. This solution, however, besides being an obvious step back, does not explain how it is possible the extensions' restriction by forcing in the definition of generic multiverse. Moreover, it cannot give an account of the way in which the generic multiverse is defined. In addition, the problem of truth reducibility remains. The difficulty with this solution is that we admit the existence of *absolutely* true propositions but these propositions are not true in the sense of the generic multiverse and at the same time we mantain the CH indetermined. The main difficulty is that any proposition φ qualitatively similar to CH can be forced to be true or to be false. We should then change the generic multiverse in such a way that permits considering certain propositions like φ, but keeps indeterminate the CH.

In conclusion, the position of the generic multiverse is, as described by Woodin, problematic in its core. The only possibility to save it is avoiding to add the Constraints and instead try to rebuild it from scratch.

2.2 Steel program

Steel's position[4] is exactly the opposite of Woodin's one. In fact, as we mentioned at the beginning of the chapter, the Steel's goal is the formalization of the multiverse so that the truth can be reduced to the truth of fragment of it, the *core* of the multiverse. So, what for Woodin was a problem to be avoided at all costs, for Steel is instead the generic multiverse strength. For Steel, it is clear that the theories belonging to the family of possible extensions of ZFC are consistent. In fact, given a theory T extending ZFC in some way, we can always construct a theory U such that $Con(U) \implies Con(T)$. Moreover, until now it has always been possible to find, by the forcing, a large cardinal hypothesis H to which a given theory T is relatively consistent. Of course, Gödel has shown that there isn't a "final theory", i.e. a theory whose consistency proves the consistency of all theories, as well as its own. However, the situation is better than it seems: we have a lot of evidence, in particular from the inner models of these cardinals, that the large cardinals hypothesis (even those particularly strong) are consistent.

Developping one of these theories means developping them all (through booleanamente evaluated interpretations): at the lowest level, that of "concrete mathematics" (for example, the theory of natural numbers) all these theories are equivalent, but start to differentiate once you climbed to the level of the transfinite, beyond ZFC. This should not discourage us: why

[4] See [Steel 2012].

developping *all* these theories should not suitable? Basically they agree on most of the mathematics that is interesting to a mathematicians that is not a set theorist, and different theories means more universes to create and embellish.

The limit of this position is that you end up having a lot of mathematics, all *fundamentally* different. But we do not want that every mathematician has his private mathematic, instead we want a shared single theory so they can use the work of others. Steel uses a botanical metaphor: if these theories are flowers, it is best that all of them flourish in a unique botanical garden, than each in a different garden. In reality the fact that all these theories are somehow connected and not independent of each other suggests that their common “concrete theory” (that is, the theory of natural numbers and real numbers) stems from deep logical relationships. These relationships are evident when analyzing the proofs of consistency related to these theory.

Thus Steel’s goal is to build a *formal* theory of the multiverse, which takes into account the individual universes (the theories of ZFC extensions) unifying the truth so that we can discuss without problems. In fact, Steel states that, in case we have two different theories, the first thing to do would be trying to unify them. In fact, the ultimate goal is that this theory can be a foundational theory in which would be possible to develop both concrete mathematics and set theory. Moreover Steel, knowing that the axioms of ZFC are not enough to solve many of the issues related to set theory and that it is not clear if there are more suitable axioms to do it, aims to build a set theory that extends ZFC in an optimal way. Steel believes that he has found in the large cardinals axioms the most natural extension of ZFC axioms, and then a theory including these axioms should be the most suitable to his objectives.

In the following pages I describe the Steel’s formalization in more detail: in the section **??** I will discuss Steel’s proposal for the axioms of the multiverse, as well as the notion of truth and the way of introducing the large cardinal within the multiverse. In the next section (section **??**) I will debate some of the most philosophical theses on the language of the multiverse. In particular, I will discuss the relationship between the multiverse and the single universe, and how we can formalize the universe as if it were a multiverse.

2.2.1 Multiverse’s axioms

The main properties of this theory is that we can *maximize its interpretative power*, i.e. we can build a language and a theory such that all existing mathematics and all future mathematics can be translated (preserving the meaning) and developed within it (in short, a foundational theory in every way). In other words, this means that an extension T of ZFC must, to be the new foundational theory, include and extend the set of provable propositions of ZFC or of any extension of it. To understand if our candidate theory T meets or not these requirements, we can consider the strength of its consistency relatively to other theories. This means linearly ordering all the theories of the form $ZFC+$ "large cardinal" taken into account in such a way that if H and T are two of these theories, then either $H \leqslant_{con} T$, or $T \leqslant_{con} H$ or, if $H \leqslant_{con} T$ and $T \leqslant_{con} H$, then $H \equiv_{con} T$. With $H \leqslant_{con} T$ we mean that $Con(T) \implies Con(H)$. So in this case we will say that, from the point of view of consistency, T is *stronger* than H (in the other two cases we shall say, respectively, that is weaker or equivalent). A consequence of this fact is that we can include a fragment of the weaker theory (if not the entire weaker theory) in the stronger theory (but this applies only to those theories, such as arithmetic, which have "natural" axioms, and not axioms as Rosser proposition). Steel's hope is that by strengthening the large cardinals hypothesis and then going up in this hierarchy of consistency, the size of the set of provable propositions will increase. At the moment, however, this conjecture is proved only for specific structures: in particular, Steel has proven it for $L(\mathbb{R})$. Within the theory, the worlds are treated as their own classes and contain sets. The end result will be a first-order theory that expands ZFC specifying which models are part of it and which are not. In particular, it will specify which worlds are "initial" and what worlds are generic extensions of these initial worlds. We can now try to formalize what has been said.

Axiom 2 (Translation Axiom)**.** *In every world W of the multiverse there exists a translation of the axioms of ZFC.*

This first axiom is pretty obvious: given an axiom φ of ZFC, in every world of the multiverse there exists an axiom φ^W that is its translation. This holds for every axiom of ZFC. From Steel's point of view this axiom is fundamental: the theory we are developping must foundational and must *enhance* ZFC with large cardinals, thus starting with ZFC axioms is mandatory.

Axiom 3 (Worlds Axiom)**.** *Every world is a proper transtive class. An object is a set if and only if it belongs to a world.*

This axiom is used to define the two variables of our multiverse. Note the difference with Hamkins' multiverse: for the latter, the worlds are *models*, while are classes for Steel. The difference is crucial: while in Hamkins each world (model) presupposed a different interpretation (and so we ended up with having as much theories as worlds), for Steel this should not happen. The interpretation of the multiverse has to be unique, like the theory that supports it.

Axiom 4 (Extensions Axiom)**.** *If W is a world and $\mathbb{P} \in W$ is a partially ordered set, then there exists a world of the form* $W[G], where G is a P-generic extension on W.$
With this axiom we define the adding of new worlds to the multiverse (there is nothing new: the main method to do this is through forcing).

Axiom 5 (Initial Worlds Axiom)**.** *If W is a world and $U = W[G]$, where G is an extension $\mathbb{P}$-generic on W, then W is a world.*

As we mentioned, in our multiverse there exist only the initial worlds and the extensions of these worlds, nothing else. In reality this is not the most precise formalization of this concept. To do so, we must consider a result of Laver and Woodin[5], which states that there is a formula ψ of LST such that if $N = W[H]$, where H is a generic extension, then ψ defines W on N from $\mathbb{P}$ and H. This result should be used to formalize the Initial Worlds Axiom: W has to be defined by ψ in U. This means that the worlds are interconnected.

Axiom 6 (Amalgamation Axiom)**.** *If U and W are both worlds, then there exist two generic extensions G and H on them such that $W[G] = U[H]$.*

Finally, this axiom states that, given two worlds, we can always find two generic extensions such that the extensions of the two world will be equivalent.

How does the truth in this multiverse work? Suppose M is a model of ZFC, G a generic extension of a world, φ a proposition in the language of the multiverse, and finally t a recursive translation function (this function translates a formula of the multiverse into a ZFC formula). In other words, $t(\varphi)$ means "φ is true in some world of the multiverse". Then we will have

$$M^G \models \varphi \iff M \models t(\varphi)$$

i.e., φ is satisfied by the model M interpreted in the extension G if and only if its translation is satisfied by M in ZFC, for every proposition φ of the multiverse. A consequence of this is that if φ is a proposition in the language of the multiverse, then MV shows that

$$\varphi \iff \text{per ogni mondo } M, t(\varphi)^M \iff \text{per qualche mondo } M, t(\varphi)^M.$$

So anything that can be expressed in the language of the multiverse can be also formulated quantifying on a single universe.

The main difference with Hamkins' broad multiverse is that many objects, for example inner models, and themselves set, are not considered as separate worlds. This however does not make the theory lose any expressive power: in fact, we can always talk about these objects *inside* the multiverse language, without complicating it with very different objects. In other words, the Steel multiverse is a class of equivalence classes based on "has the same information". For the same reason, as in Woodin's case, we do not admit generic extensions on classes. In fact, if we started to consider sets, inner models and generic extensions on classes, the Amalgamation Axiom would no longer be satisfied.

[5] See Lemma 33 in [Woodin 2009].

As we mentioned at the beginning, one of the main concerns of Steel is to preserve, in the multiverse, the large cardinals. The theory MV formalized above allows us to introduce large cardinals hypotheses as follows: given a large cardinals hypothesis φ, we add to MV a proposition of the form "for all the worlds W, φ^W holds", where φ^W is a translation of φ in MV language. For example, we can add "for every world W, (there is$asupercompactcardinal)^{W''}.Thiswouldimplythat, fromthetheoremoflargecardinalspr$
for any φ in the theory that they generate.

2.2.2 Some thesis on the multiverse language

In this section I will talk about the relationships between the standard first order language of set theory (that is the language based on a single universe) and the multiverse language that we have just formalized.

In fact, we can consider the standard language of set theory as the language of the multiverse *with* a constant symbol V for the universe of reference. Propositions like the CH are considered statements about the universe of reference. If a setting like this makes us lose some expressive power is an open question. We can explain three possible solutions.

Thesis (Weak Relativism)**.** *Every proposition that can be expressed in the standard language LST can be expressed in the language of the multiverse.*

From this thesis it follows that the symbol V makes sense if and only if we can define it in the language of the multiverse, but says nothing on the actual ability to do so. The main argument in favor of this thesis is that all of mathematics developed so far can be expressed in the language of the multiverse. Maybe we lose something in doing so (for example something about V that does not involve the definition of V in the multiverse), but at the moment it's hard to figure out what. Moreover, the thesis seems perfectly in line with the Translation Axiom and the way we translate the large cardinals in the language of the multiverse.

Thesis (Strong Absolutism)**.** *V makes sense and is* not *expressible in the language of the multiverse.*

We can consider this view the opposite of the previous one. One argument in favor of this thesis is that the language of the multiverse is based only on the standard language. In fact, our unique knowledge about the multiverse come to us through the translation function t. The description of M^G, for example, shows only how to translate the multiverse in the standard language, but says nothing about the meaning of the multiverse language. We can respond to this argument, claiming that the meaning of the language of a foundational theory is given by its use and, in particular, by the theories that are developed inside it. From this point of view, the language of the multiverse has been used for a long time and almost all of set theory has been developed in it. This position considers set theory's model theory naturally connected to the multiverse, since in the formal semantics of set theory (its model theory, to be precise) one of the main topics is translating a model into another. So the language of the multiverse only serves to isolate the important parts of the standard language, i.e. the image of the function t, eliminating the parts without meaning, to avoid meaningless questions. After this work of "debugging", we can totally eliminate the standard language LST, which has helped us defining MV, to use only the latter, a more powerful and improved version of LST.

Finally, it is possible to formulate a moderate thesis, combining the idea of weak relativism thesis with the idea of a distinct unique world of reference of the strong absolutism thesis. Following this path, we will have a single world definable in the language of the multiverse. By forcing it can be shown that if the multiverse has a definable world, then this world is unique and is included in all other worlds. We call this world the *core* of the multiverse.

Thesis (Weak Absolutism)**.** *There is a unique world that is definable in the language of the multiverse (that is, the multiverse has a core).*

This view is fundamental to the future development of the multiverse: in fact, if it were shown that the multiverse actually has a core, this core would be important and worth studying, either if it is V or it is some another world. At present, however, there is no way to answer this question: neither MV nor its extensions with large cardinals can decide the issue. We cannot even formalize a theory of this core, granted the existence of it. Simply put, we are in the same situation of the Continuum Hypothesis: would be of great help a positive answer to the question, but we cannot give one at all. But, unlike the CH, in this case there is no independence proof of the issue, then at least a bit of hope is still there.

Concludiamo citando una proposta di Woodin [6] per un assioma che sarebbe molto utile nel tentare di risolvere questi problemi. L'assioma proposto da Woodin infatti dovrebbe

- implicare che il multiverso abbia un nucleo;
- proporre una "teoria dalla struttura fine" per questo nucleo;
- essere consistente con tutte le nostre ipotesi sui grandi cardinali.

Gli strumenti teorici richiesti, anche solo per sondare questa possibilità, sono molto avanzati (in alcuni casi non esistono ancora!), ma sicuramente lo studio di questo assioma e delle sue possibili conseguenze sarà uno dei programmi di ricerca più interessanti nei prossimi anni.

In conclusion, we have to quote a proposal made by Woodin[7] for a very useful axiom. This axiom should satisfy the following requirements:

[6] Ancora non pubblicata, ne discute [Steel 2012].

[7] Still unpublished, in [Steel 2012] there is a detailed discussion about it.

- it should imply the existence of the core of the multiverse;
- it should imply a natural theory for this core;
- it should be consistent with all the large cardinals hypothesis that are known.

The theoretical tools required only to explore this possibility, are very advanced (in some cases do not exist yet!), But surely the study of this axiom and its possible consequences will be one of the most interesting research programs in the upcoming years.

Capitolo 3

The Hyperverse and the Vertical Multiverse

In this chapter I will describe the conception of the “vertical” multiverse. According to this position, is possible to enlarge the universe V in “height” (that is, we can add new ordinals) mantaining its “width” unchanged (that is, without adding new subsets).

More precisely, the position now described is just one of the possible ones: for example, we can imagine an “horizontal” multiverse (in which we add new subsets of V without adding new ordinals). In section 3.2 I will describe this alternatives, while in section 3.1 I will discuss the Hyperverse Program. With hyperverse we denote the whole collection of transitive models of ZFC, that will form the machinary needed to apply model theoretic techniques and forcing (in particolar the Type Ommission Theorem), with the ultimate goal of studying the standard universe V. The last section of the chapter (section 3.2.1) will be about V-logic, that is an interpretation of the hyperverse in the vertical multiverse.

The vertical multiverse is much more close to an anti-pluralist position that defends the Single Universe than to the pluralist conception of the multiverse (we can say the same for Woodin and Steel, the *formally* develop multiverses, but these multiverses are reduced to a Single Universe), but can still be considered from the pluralist point of view. In fact, the main example of generic multiverse, the conception of Zermelo’s *natural domains* can be considered a collection of universes. Thus the vertical multiverse is in between multiverse and Single Universe: although cannot be consider a true multiverse, shares with it some important properties.

3.1 The Hyperverse

The Hyperverse program, as it has been defined by Arrigoni and Friedman [1], try to decide which propositions of first order ZFC are true in V. The fact that we speak about the universe V should not deceive us: even though this approach is, in fact, an anti-pluralist approach (dealing with a single universe), methods and tools are those of the multiverse. In fact, in order to study the properties of V, Arrigoni and Friedman built up a context composed of various universes, where each universe appears to be a different image of V. In other words, we have to imagine that V has certain properties, to see the consequences of each of these "image" of V. We call this context *hyperverse*, and we define it as the collection of all the transitive models of ZFC.

That said, one of the most important issues (as in all other multiverse) is to decide which universes can be part pf the hyperverse and which not. We will therefore find some criteria, based on the comparison of the various universes, to prefer some universes to others, and also be able to justify this preference. This captures the essence of the hyperverse as "half-way" between the pluralistic multiverse and the anti-pluralistic universe: the question is evidently typical of the multiverse (decide which universes will be part of the multiverse and which are not), but the fact the choice must be justified is a problem that interests only an anti-pluralist (for a pluralist, the only justification is that some universes are preferred for practical reasons). Such an approach raises the question of how to harmonize discordant criteria: for example, some may prefer universes where the assumptions for large cardinals are met, while others might prefer universes in which Martin's axiom is satisfied. So we must also define the principles that make the search for these two " harmonic " criteria: the principles are the principle of *maximality* (i.e. the principle that every universe must be as large as possible) and the principle of *omniscience* (i.e. every universe must be capable of describing what is going on in the other universes).

The ultimate goal is to find propositions that can be added to the "new" set theory as axioms. These axioms are those propositions which are true in all universes chosen as part of the hyperverse, hoping that they also include the answer to some question still independent.

Nelle sezioni seguenti tratterò più nei dettagli del programma dell'iperverso. Per prima cosa, nella sezione 3.1.1, verrà discussa la nozione di verità dell'iperverso e la sua essenza antiplatonica. Nelle due sezioni successive invece procederò nella formalizzazione dell'iperverso. Nella sezione 3.1.2 epliciterò le caratteristiche principali che l'iperverso deve avere, nonché come si relaziona con V. Concluderò con il tentativo di formalizzare i criteri di preferenza dei singoli universi (sezione 3.1.3).

[1]See [Arrigoni & Friedman 2013].

3.1.1 The anti-platonic notion of truth of the hyperverse

The conception of the hyperverse is not a platonic one: universes taken from time to time into consideration do not actually exist (as was the case in Hamkins' multiverse). Moreover, platonism is not invoked regarding V. In fact, in the search for solutions to independent issues, it is never taken into account the very specific reality as an argument in favor of the legitimacy of some solutions over other ones.

This anti-platonism is reflected in the notion of truth of the hyperverse:in fact, saying that a statement is "true for V" doesn't mean that we are saying something about the ontological status of that proposition within the universe of set theory, but simply describes the status of certain propositions in a particular axiom system. The reality of this axiom system does not exist independently from the practice of set theory. But then what are, for the hyperverse, these propositions "true in V"? Since one of the hyperverse's main problems is the choice of the universes, we can say that the "true propositions in V" are those propositions which are, or are to be regarded as, definitive, i.e. not subject to revision. These definitive propositions are of two types:

- the *de facto* truths of set theory;
- the *de jure* truths of set theory.

The first are those propositions which, for the role they play in the daily practice of set theory and mathematics, are not to be contradicted by new axioms of set theory. For example, the axioms of ZFC and the consistency of $ZFC+$ "large cardinals axioms" are truths of this kind. The de jure truth instead are those propositions which not only does not contradict the de facto truths, but are true in all universes chosen to be part of the hyperverse. Consider that the de facto truths must not necessarily be true in all universes: for example, are part of the de facto truths not only the axioms of ZFC, but also those for $ZF + AD$, or those for Z_2. Each of these systems of de facto truths defines a set of different universes: the task of the hyperverse program is to understand which of these systems is preferable for the construction of the hyperverse and for the investigation of the independent issues. The formulation of the de jure truths is not based on external constraints, such as loyalty to a reality that we try to describe (we said that such a reality does not exist), but is based solely on mathematically justifiable procedures within the system.

At this point is clear how the hyperverse is not a fixed and independent reality, but only a mathematical construction, developped as an instrument to investigate set theory.

For Arrigoni and Friedman the research of de jure truths must not be influenced by any personal beliefs about how these truths have to solve certain independent issues nor certain assumptions must be invoked as criteria for the selection of the universes. For example, preferring a certain set of universes in which the axiom of constructability ($V = L$) is a de jure truth to positively resolve the question of CH would be wrong. For them the hyperverse program does not target the solution of certain issues independent from ZFC, or the study of certain areas of mathematics. The development of set theory that go beyond ZFC should not be treated as de facto truths, but must enter the competition to be de jure truths. So the principles and the criteria by which the universes of the hyperverse are chosen must be based only on the analysis of the most general properties of ZFC.

3.1.2 The hyperverse and its relation with V

Wanting to make explicit the hyperverse characteristics, we can consider it as an attempt to arrive at new de jure truths of set theory starting from a multiverse which summarizes all the results obtained so far in set theory. Of all the multiverse possible, the best to start with is one that focuses on the well-founded models of ZFC. In this way we will have that the axioms of ZFC are de facto truths of set theory and that the universes that will be part of the hyperverse will all be well-founded. We can then define the first characteristic of the hyperverse:

First Characteristic The hyperverse must be as rich as possibile, but must not be ill-founded nor absolutely infinite.

This first characteristic implies that the hyperverse meets a criterion of *maximality* (must be as large as possible) and a policy of *well-foundedness* (so no sets that are members of themselves are allowed). Thus the set of all countable transitive models of ZFC satisfies this first characteristic. Regarding the hyperverse's size, this is guaranteed by the fact that the first characteristic implies the closure of the hperverse for all adding operations of universes (forcing on classes, forcing on sets), then the hyperverse will contain generic extensions of sets, classes and basic models.

Since the first characteristic requires that the hyperverse is formulated in a mathematically precise way, it must be possible to implement the selection of universes in in it. This would be impossible in the case of ill-founded or without end hyperverses. To do this, we introduce a second feature:

Second Characteristic In the hyperverse is possible to expresse a preference for some of its members following some principles previously justified.

As we have said in the previous section, the first order propositions that are true in V are true in all the preferred universes:

Third Characteristic Every first order property of V is reflected in a transitive model of ZFC that is a preferred universe of the hyperverse.

A consequence of this feature is that, while the criteria for the selection of favorite universes may not be first order, the de jure truths we're going to formulate are first order propositions. This proposition will be satisfied in all the chosen universes. To justify this principle we can refer to the Downward Löwenheim-Skolem Theorem in the version applicable to classes, although this, to be precise, simply implies that there must be a member of the hyperverse that reflects V. Then that this universe is one of the favorite ones is a further hypothesis of the hyperverse program, based on the fact that in this way we can enlarge the set of truths of set theory in a reasonable manner. In any case, the theorem allows universes of the hyperverse to communicate first order informations of V.

In conclusion, these features what strategy suggest in the search for new truths of set theory? First, we start from the hyperverse that more closely reflects the possible images of the universe of set theory (for example, a hyperverse might be based on ZFC, or on $ZF + AD$). Since, however, the image of V given from the entire hyperverse may be too large, or too confusing (for example, in the hyperverse based on ZFC we will have universes in which $ZFC + CH$ holds and universes in which $ZFC + \neg CH$ holds), the two characteristics described above (features (2) and (3)) allow us to make a choice between the various universes, to keep only the universes that possess the properties that obey the criteria that we are going to explain in the next section. These preferred universes will have a shared set of truths (de jure truths), from which we will choose the truths that we are going to add to the axioms of set theory, to enrich its set of truths.

3.1.3 Criteri per la preferenza degli universi

The problem is now all in the choice of these favorite universes. We have already mentioned that the criteria for this choice should be the most possible free from any influence by certain areas of set theory (criteria therefore useful only to the development, for example, the program of the inner models programs are not good), or of particular areas of mathematics (i.e., criteria to investigate, for example, certain algebras will not be chosen). We also said that we should not choose to be such criteria hypothesis or particular conjectures. Some examples of this non criteria are:

- The Generalized Continuum Hypothesis (GCH);
- $V = L$, despite its combinatorial strenght;
- Projective Determinacy (PD), that implies an interesting set theory of projective sets of reals;
- Forcing Axioms (like PFA), despite theirs combinatorial strenght.

All these non criteria meet special needs of certain areas of set theory, so they cannot be chosen as criteria to decide which universes to take into consideration and which not. Obviously, *after* the selection of these universes, all the previous propositions can (and must) compete to be new de jure truths, and in some cases (for example the GCH) would be particularly helpful if some of them are accepted as de jure truths.

Moreover, all the criteria must also not contradict the de facto truths (i.e., in our case, the axioms of ZFC) of the hyperverse. For example, let's say that one of the criteria for the selection of the universes is the *minimality*: the chosen universes must be as small as possible. A criterion like this can lead to the choice of a single universe, the *minimal model* of ZFC. But this would imply that the proposition "there are no models of ZFC" is a property of V. But this is in contradiction with the current practice of set theory, in which the models of ZFC exist and this is part of the de facto truths of set theory. Even a weaker criterion, such as the one that we must prefer universes that satisfy $V = L$, will face the same problem. In fact, despite $V = L$ allows the existence of ZFC models, it does not allow the existence of inner models of ZFC with measurable cardinals. But, as in the previous case, for the current practice of set theory these models exist, and are part of the de facto truths of the theory.

The criteria that we are going to consider are those of *maximality* and *omniscience*. We start by analyzing the maximality. First, we note that it is not possible to have, in the hyperverse, a "structural maximality", that is we cannot have a universe that contains all possible ordinals. This is because there is a no countable transitive model of ZFC greater than all (remember that we defined the hyperverse as the collection of all countable transitive models of ZFC): we can in fact, given any model, always add new ordinal to build a larger model.

So we can now state the first principle:

Logic Maximality Let v be a variable on the elements of the hyperverse. v is *logically maximal* if and only if all the proposition of set theory with certain parameters that hold "externally" (so in some universe that contains v as a sub-universe) also hold "internally" (that is, in some sub-universe of v).

From this very general principle we can formulate two criteria, based on which parameter we take into consideration for the propositions of set theory:

Ordinal Maximality Criterion We define the universe w a *lenghtening* of v if and only if it's an initial segment of w. v is maximal in respect to the ordinals if and only if it has a lenghtening w such that for every first order formula φ and for all subsets $A \subset v$ belonging to w, if $\varphi(A)$ is satisfied in w then $\varphi(A \cap v_\alpha)$ is true in v_β for a couple of ordinals $\alpha < \beta$ in v (here with v_α we denote the collection of set belonging to v with rank less than α);

Power Set Maximality Criterion If a proposition without parameters is true in some outer model of v (that is, in some universe w containing v and with the same ordinals than v), then is also true in some inner model of v (that is in a universe v_0 contained in v with the same ordinals than v).

This two criteria are complementary: while the first one requires that the models have the power set operation fixed (that is, we cannot use it to enlarge the models), the latter criterion requires instead that the ordinals are fixed.

The criterion of ordinal maximality is usually known as *principle of higher-order reflection*, and implies the existence of large cardinals consistent with $V = L$ (for example, the inaccessibles cardinals, weakly compact cardinals, ω - Erdős, etc.). On the contrary, the power set maximality criterion is much more recent: it is equivalent to the hypothesis of inner models (IMH). In informal terms, this hypothesis states that giong from v to one of external model does not change its *internal consistency*, that is the set of sentences without parameters in an inner model of v remains unchanged. The problem is that the IMH refutes the existence of inaccessible cardinals: so,how can we make the power set maximality criterion compatible with the de facto truths of set theory? But if we consider the large cardinals as existing *only* in the inner models, and not in V, then the problem does not arise: in fact, the IMH is compatible with the existence of large cardinals in the inner models! At this point, we have two contradictory criteria. The best course of action is combining them in a single consistent criterion (satisfied by at least a universe of the hyperverse). We can formulate the following conjecture:

Conjecture 1 (Syntetize Maximality)**.** *We define the* power set maximality* *(IMH^*) as the power set maximality (IMH) reduced to the maximal universes in respect to the ordinals (that is, the universe in which a true statement for an outer model of v is true also for its inner model). Then the conjuction of the ordinal maximality criterion and the power set maximality criterion is consistent. In other words, there exist universe that satisfy* both *criteria.*

The proof of these conjecture requires the same methods used to prove the consistency of the IMH and the application of Jensen Codification Theorem to measurable cardinals.[2]

Is possible to formulate another criterion to help us choosing preferred universes: the *omniscience* criterion. A universe is omniscience if and only if can describe what can be true in the other universes:

Onmiscience Criterion Lets Φ be the set of propositions with arbitrary parameters from v than can be satisfied in some outer models of v. Then Φ is first order definable in v.

[2]See [Friedman&Welch&Woodin 2008] for details.

We can try to synthesize this criterion with the criterion of ordinal maximality, while a synthesis of all three criteria seems much more difficult to achieve. In fact, the first approach that comes to mind, affirm the maximality of the power set of omniscient and maximal universes as regards to ordinals, is inconsistent.

In conclusione, questi criteri riescono a sviluppare molto bene la strategia del programma dell'iperverso che abbiamo esplicitato nella sezione precedente. L'unico problema, a mio avviso, è che possano risultare troppo restrittivi nella scelta degli universi, portando a lasciare senza soluzione la maggior parte delle questioni indipendenti che ci interessano (come la CH).

In conclusion, these criteria are able to develop very well the strategy of the hyperverse program we have explained in the previous section. The only problem, in my opinion, is that they may be too restrictive in the choice of universes, leading to leave us without a solution to the majority of independent issues that interest us (like the CH).

3.2 Actualism and Potentialism

Before defining actualism and potentialism (and theirs ramification, if any), we have to take into consideration the various operation capable of enlarging V. There are two possibilities: we can either *widen* V adding subsets, or we can add new ordinals (e.g. considering V_{ε_0}).

First of all, lets consider the actualist position. For an actualist, it is not possibile to enlarge V, neither adding new ordinals nor adding subsets. As a matter of fact, any construction from those methods seems to be an extension of V, but, in reality, is just a model *in* V. This position is naturally compatible with anti-pluralism and the existence of only one universe. But, surprising, it is not incompatible with the multiverse. For example, even Hamkins' multiverse, the most extreme among the pluralist universes, can be defended by an actualist: indeed, Hamkins' multiverse doesn't need anything else than V, and doesn't need any extension of V neither.[3] We can distinguish three kinds of actualism, based on what operation (adding ordinals or adding subsets) is forbidden:

Absolute Actualism : Both height and width of V are fixed (that is, we can neither add new ordinals, nor subsets);

Ordinal Actualism : The height of V is fixed (we can add only new subsets, not new ordinals);

[3] In the whole chapter with *extension* of V we mean operations like taking V_κ, where κ is, for example, a Woodin cardinal, and not operations like the forcing extension $V[G]$. In fact, in the first case is exactly V that is extended, while in the latter case V remains unchanged.

Power Actualism : The width of V is fixed (we can add only new ordinals, not new subsets).

Quite the opposite is the position defended by the potentialist: for him, V is an undefined object, and not a fixed entity. Obviously, some of the properties of V are fixed and stable, but, overall, V can always be modified adding new ordinals or adding new subsets. As in the latter case, we can divide this position in three sub-positions:

Absolute Potentialism : Neither height nor width are fixed, so we can add to V both new subsets and new ordinals;

Ordinal Potentialism : The height of V is not fixed (we can add new ordinals, not new subsets);

Power Potentialism : The width of V is not fixed (we can add new subsets, not new ordinals).

Is pretty easy to spot a parallelism between the two "moderate" forms of actualism and potentialism. In fact, the Ordinal Potentialism is equivalent to the Power Actualism, while Ordinal Actualism is equivalent to Power Potentialism. Despite this, there are still differences between, e.g., an Ordinal Potentialist and a Power Actualist. The main reason is that an actualism will be much more inclined to realism (V is unmodifiable because it exists somewhere) than the potentialist (considering V modifiable undermine its actual existence). Another thing worth noting is that moderate potentialsim and moderate actualism are ordinal maximality and power maximality (that we introduced for the hyperverse) generalized to the whole universe V.

An example of "moderate potentialism/actualism" is [Zermelo 1930]. In that article Zermelo proved that the axiom of second order set theory, Z_2, are *quasi-categorical.* In other words, every model M of Z_2 is of the form $\langle V_\kappa, \in \rangle$, where κ is a strongly inaccessible cardinal. These V_κ are called by Zermelo *natural domains*.[4] The sequence of these domains is dynamically build, as if the universe would unfold accordingly.

This unfolding can be explained as a continuous actualization of the universe. Up to where is unfolded the universe is an actual one, thus unmodifiable. This universe cannot be extended adding new subsets, so we can consider Zermelo's construction a power actualism. Moreover, since Zermelo's theory is a second order one (for Zermelo there exists a set of properties on which the axioms, and the Axiom of Separation in particular, can quantify), the powe rset operation has to be fixed (another feature of

[4] For historic completness we have to note that the terminology used by Zermelo back in 1930 was a bit different: e.g., for Zermelo a strongly inaccessible cardinal was a "exorbitant number". To ease the reading in this chapter the terminology and some of the concepts has been modernize. Another difference is that V, in Zermelo's theory, was based upon atoms, thus the width was fixed in its base, since it wasn't possible to add new atoms.

power actualism). But every natural domain is extendible in height, simply by taking the next domain. In fact every domain is indexed by inaccessible cardinals: given a domain V_α, we can extend it to a domain $V_{\alpha+1}$, in this way adding a new ordinal. In other words, if we have a proper class of inaccessible cardinals, every natural domain can be extended to a higher one. So the concept of the universe described by Zermelo can be also considered an ordinal potentialism

A questo punto, se analizziamo più in dettaglio la costruzione di Zermelo risulta chiaro come i domini naturali possono essere considerati un *multiverso verticale*, in cui ogni singolo universo è indicizzato da un cardinale inaccessibile. Infatti, secondo questo punto di vista, l'universo V altro non è che una collezione di V_α, la cui larghezza è definita mentre l'altezza complessiva è estendibile (ossia, l'altezza dei singoli V_α è, ovviamente, fissata, mentre sarà sempre possibile considerare un $V_{\alpha+1}$).

At this point if we analyze in more detail Zermelo's construction it will be clear that the natural domains can be considered a *vertical multiverse*, in which every single universe is indexed by an inaccessible cardinal. Indeed with this point of view the universe V is a collection of V_α with a fixed width and an extendible height (that is, the height of every single V_α is fixed, while it is always possible to take a $\mathbb{V}_{\alpha+1}$).

3.2.1 The Hyperverse and the Vertical Multiverse

In this section I will show how the hyperverse is compatible with Zermelo's vertical multiverse introduced in the previous section.

At first, hyperverse and vertical multiverse seem totally incompatible: while the first allows the possibility of adding new subsets by the means of the power set, instead in the vertical multiverse this is not possible. But, using the V-logic, we can express the properties of the enlargment of V without implying the existence of this enlargement. Also, these properties will be first order properties on $Hyp(V)$, that is a modest lenghtening of V. This mathematical machinary allows us to avoid violating Zermelo's conceptions of the vertical multiverse.[5]

First of all we need some details about the infinitary logic $\mathcal{L}_{\kappa,\omega}$, where κ is a regular cardinal (V-logic is a special case of this logic). The language of this logic will be composed by

- κ variables;
- up to κ constants;
- the symbols $\in, =$;
- auxiliars symbols.

[5]For mathamatical details see [Barwise 1975].

The formulas of $\mathcal{L}_{\kappa,\omega}$ are defined by induction:

1. Every first order formula is in $\mathcal{L}_{\kappa,\omega}$;
2. if $\{ \varphi \}_{i<\mu}, \mu < \kappa$ is a set of formulas in $\mathcal{L}_{\kappa,\omega}$ such that there are finite variables in these formulas, then the infinite conjuction $\bigwedge_{i<\mu} \varphi_i$ and the infinite disjunction $\bigvee_{i<\mu} \varphi_i$ are formulas in $\mathcal{L}_{\kappa,\omega}$;
3. if φ is a formula of $\mathcal{L}_{\kappa,\omega}$, then its negation and its universal closure are also formulas in $\mathcal{L}_{\kappa,\omega}$.

In [Barwise 1975] is developped the notion of provability in $\mathcal{L}_{\kappa,\omega}$, and it's proved that the syntax is complete, in regards to semantic, when $\kappa = \omega_1$.

As we already said, V-logic is a specific case of $\mathcal{L}_{\kappa,\omega}$. Suppose that V is a transitive set of cardinality κ and consider the logic $\mathrm{L}_{\kappa^+,\omega}$ extended with κ variables $\{ \bar{\alpha}_i \}_{i<\kappa}$ for all the elements α_i in V. In this logic we can form an infinite formula that guarantees that if M is a model of this formula (moreover, the formula will define some desirable properties of M), then M is an outer model of V (that satisfies the properties defined by the formula). If the set V is countable and the formula is consistent, then the model M exists in the universe we are considering. Instead, if the set V is uncountable, then there are no guearantees that the model exists (but in this case we can avoid any interpretation and stick to the syntax of the formula).

Before stating the main result of the section, we have to define the *admissible set*. M is an admissable set if and only if is a model of a weak fragment of ZFC. In particular, if it is a model of the Kripke-Platek theory (ZF without the Power Set Axiom and with weaker versions of the Separation Axiom and the Replacement Axiom). Without ruling out every detail of this theory, we need only to point out that in this theory, for any set N exists the minimal admissable set M such that $N \in M$ and M is of the form $L_\alpha(N)$ for the least ordinal α such that M satisfies the theory KP. We denote all of this with $Hyp(N)$.

Theorem 8 (Barwise)**.** *Lets V be a transitive set model of ZFC and $T \in V$ a first order theory that extends ZFC in the hyperverse sense. Then there exists an infinite formula $\varphi_{T,V}$ of the V-logic such that all of the following is equivalent:*

1. *$\varphi_{T,V}$ is consistent;*
2. *$Hyp(V) \models$ "$\varphi_{T,V}$ is consistent";*
3. *If V is countable, then there exists an outer model M of V that satisfies T.*

This theorem allows us to consider outer models of V (that are the enlargement of V) in $Hyp(V)$. The latter is a modest lenghtening of V that doesn't widen V.

In conclusion, we can consider $Hyp(V)$ as an “image” of the hyperverse: in fact in $Hyp(V)$ we can behave just like in the hyperverse, but without requiring the satisfaction of the Power Set Maximality Criterion. Moreover, in this configuration is possible to armonize the maximality criterions with the Omniscience Criterion. In fact, while Ordinal Maximality is perfectly compatible with omniscience, we have some problems introducing also Power Set Maximality. But, if we consider the hyperverse as $Hyp(V)$, we will end with a hyperverse compatible with the vertical multiverse. Lastly, we can even strenghten our choice criterions for the universes (e.g. a universe to be chosen must satisfie both Ordinal Maximality and Omiscience Maximality).

Capitolo 4

Martin's position: the One Universe

In questo capitolo tratterò della posizione antipluralista in teoria degli insiemi. Come nel caso del pluralismo, anche per l'antipluralista il problema risiede nei risultati d'indipendenza. A differenza che nel caso del pluralista, per l'antipluralista questi indicano solamente che l'attuale sistema di assiomi non è sufficiente per giustificare tutti gli enunciati matematici. Quindi, secondo questo punto di vista, bisogna cercare nuovi assiomi che migliorino il sistema originale, affinché questo possa giustificare un numero maggiore di enunciati. Questi nuovi assiomi devono essere giustificati a livello teoretico, e non solo a livello pratico. Tutto questo deriva del fatto che (o, in alcune varianti, implica) esiste un mondo matematico reale e oggettivo (quindi è una posizione inerentemente platonica).

In this chapter I will discuss the anti-pluralist position in set theory. As in the case of pluralism, even for the anti-pluralism the problem lies in the independence results. Unlike in pluralism case, for the antipluralism advocate these results only indicates that the current axiom system is not enough to justify all mathematical statements. So according to this view, we must seek new axioms that improve the original system, so that it can prove a larger number of propositions. These new axioms must be justified at a theoretical level, and not just on a practical level. All of this derive from the fact that (or, in some variants, implies) there is a real and objective mathematical world (thus is inherently a Platonic position).

This chapter I will discuss the more traditional position in set theory. In fact, for most of the set theorists the universe of set theory is *unique*. This position has been clarified perfectly by [Martin 2001], who is also the most vigorous proponent. Martin's arguments derive from [Zermelo 1930], but his position is not strictly based on it. In fact, while Zermelo's construction can be considered a multiverse (see **??**), Martin's conception is the classic ZFC considered in V. But Martin bases his work mainly on categorical results to argue against Field and Balaguer (cfr. chapter **??**), that the existence of multiple universes that satisfy ZFC is impossible.

But note that Martin's goal is not to prove that all the propositions of set theory have a fixed truth value: for him the question still remains open, and for him a negative answer is still possible. Instead, his arguments want to show that, either the usual concept of set is more than accurate and suitable for the formalization of set theory (thus eliminating the need for alternative concepts in alternative universes), or the notion of truth to which the approaches of the multiverse recall it is inherently wrong (and then, either we can find a new satosfying concept of truth, or we cannot defend pluralistic thesis).

The following sections are dedicated to these arguments. After discussing the concept of set that we're going to take into consideration (section **??**), in the next section (section **??**) will I report the main argumentations that Martin uses against the multiverse, that are based on the categoricity results.

4.1 The concept of set

First we need to consider the concept of set. For Martin, the concept of set is the *iterative* concept: the sets are formed in a transfinite process, beginning with a non-empty domain of "non-sets" (for Martin these are the *urelements*) . If this concept fully determines which objects are sets and which are not, then it is defined in a satisfactory manner. In trying to understand if the iterative concept of set is defined in a precise manner or not, we cannot take the most direct route: any response would refer to further questions (for example, "the concept of membership is well defined or not?") or would not give conclusive answers. In fact, as a negative response on the definiteness of the concept of natural number it can not solve everything concerning the concept of number (stating that the concept of natural number is not well defined tells us nothing about the real numbers, for example), thus the same negative response to our question would be compatible with the concept of categoricity, and then "well-defined enough" to allow us to deduce the truth values.

So we have to take the longer route, dividing our question ("is the iterative concept of set well defined?") in three parts:

1. Given any objects, what *object* is the set of these objects?

2. At any new level of the transitive iterative process, what new sets are formed from the older ones?

3. How are ordered the levels of this process? In particular, how long it is this process?

The iterative concept of set will be "maximally defined" if and only if it will be possible to answer all these questions, and will be inherently vague if you cannot give an answer to any of this questions. To those who defend the multiverse, the case is the latter: since the concept of set is vague and cannot be defined, we can change it at will, to build more and different universes. For Martin (and all the anti-pluralist) this is not possible: since the iterative concept of set is well defined, any change must be profoundly justified on solid foundations. This justification, however, must be so deep that the old concept will result inappropriate to the new purposes, thus becoming obsolete. So the universe will always be unique.

To answer the first question, lets suppose we have two objects x and y. Is there a *unique* object that is the set $\{ x, y \}$? In other words, avoiding ontological assumptions, is determined what the object $\{ x, y \}$ is to be guarented that this object is unique? Moreover, is possible to fix this object?

A way to resolve all these problems is assuming the *Uniqueness Postulate*:

1. the sets are determined only by their elements (this is the extensionality principle);

2. the elements of a set are determined by the set itself (this is the base of the separation principle).

We can understand these statements in various ways. The most obvious one is not allowing *in any way* that two objects of the form $\{ x, y \}$ exist. This, however, collides with many results of model theory. Then the more correct interpretation states that, although there might be many structures with a set $\{ x, y \}$ inside them, inside a single structure there can be only *one*$\{ x, y \}$. This interpretation is essentially the classic one: within the single structure we apply extensionality, while the uniqueness postulate is appied between different structures. The first part of the latter implies that different structures that satisfy the same concept of set must have the same $\{ x, y \}$, while part (2) states that one set cannot have different members even in different structures (so our set cannot be $\{ x, y \}$ in a structure and $\{ x, y' \}$ in another).

Although assuming the Uniqueness Postulate makes it very easy to define the concept of set, Martin prefers to avoid it. The reasons are multiple: first, proponents of the multiverse (particularly Field and Balaguer) tend not to assume it (because this would limit the number of possible structures), and the Martin's goal is a confrontation as equal as possible. Secondly, the postulate is also denied by any structuralist: the position that any object can be considered a set and only the isomorphisms count, although it is quite rare (even more rare than structuralism regarding the numbers), it is nevertheless valid. In fact, Martin has no argument against such a position, besides the uniqueness postulate, which would make it impossible. However, Martin prefers to allow the structuralist position than asserting a postulate of which, in any case, is not totally sure. Finally the last problem that Martin sees in the Uniqueness Postulate is that makes things "too" easy, almost in a suspicious way.

But without the postulate the answer the question on the definiteness of the concept of set becomes very difficult. In fact, the postulate of uniqueness and the concept of membership, together, are enough to define which objects belong to which sets. If we eliminate the postulate, the answer to a question like that becomes much more tortuous. In any case, even without the postulate, it remains possible to determine the truth value of any statement of set theory, then, for our goals (and those of Martin), we can safely avoid assuming it.

At this point we can formulate three different answer to question (1):

- the concept of set is determined by the Uniqueness Postulate and by Extensionality;
- the concept of set is determined by the various possible isomoprhisms between objects (structuralism);
- there exist various concepts of set, but in the same structure only one concept at a time is allowed.

For a pluralistic this would be more than enough: we have three different concepts of set that correspond to as many different universes. Instead for Martin (and for the anti-pluralist in general) these three conceptions are competing with each other: the correct answer is one and only one.

Concerning the questions (2) and (3) the situation is the following. The answers are going to configure two different concepts of set: the *strong* iterative concept, which implies the existence of the set considered (a platonist approach), and the *weak* iterative concept, which does not imply the existence of the set, but describes the process by which the sets that exist are formed and, in part, their nature. We can also define a *weaker* weak concept, which instead requires no ordering. According to this concept iterative sets are simply those objects obtained from the urelements forming all sets that can be formed by urelements and sets that have already been formed. The ordering of this formation is well-founded, but may be partial and does not necessarily correspond to a linear ordering of the levels. The answer to questions (2) and (3) therefore varies with the concept taken into account (we will not consider the weaker weak concept, because the answers are negative for both questions).

According to the weak concept, a possible answer to the second question might be the following: at the level s are formed *all* sets that have not been formed before the level s, but whose members have all been already formed in an earlier level. Note that such a response does not necessarily imply that at a given level x some sets are actually formed, but since the iterative process would stop if this does not happen, we assume that at each level is formed at least one set. Instead, according to the strong concept, the answer is more complex. For example, we could respond by saying that every defined property P determines a set that is formed at the level s if not before: the set of all sets before level s with the property P. This response, however, is likely to be vague, especially for the skeptic. But Zermelo proved that his Axiom of Separation, which reformulates this in a more indirect manner, is precise enough and has enough demonstrative force. Its ontological implications, however, are considerably weaker.

The third question requires that the levels of the iterative process are well-ordered, so any answer must include this fact. Not to mention that we may have doubts even on the definiteness of the concept of well ordering[1], a consequence of this is that we start the iterative process with an ordered sequence as an initial segment of the natural numbers, and this segment, in the case of the weak concept of the iterative process, can be proper. In addition to answer the question we must also give an account of how long is the iterative process, i.e. we have to specify the length of the well-ordered sequence of levels. The answer, as with the previous question, changes according to the iterative concept that it is considered. For the weak iterative concept, the answer is simple: the iteration continues until new sets are formed, and it ends when there are no new set formated whose members have already been constructed. For this answer the strong iterative concept is in trouble: one option is to appeal to the concept of *absolute* infinite (which is supposed to be bigger than the ordinary infinite), and then say that there is an absolutely infinite number of levels. In addition, the Axiom of Infinite and the Supersede Axiom can both be seen as consequences of the absolutely infinite number of levels.

At this point we can define what it means, for a structure (that Martin does not define in a formal way, but leaves deliberately as an informal concept and therefore more flexible), to satisfy a concept of set. For Martin a structure is, approximately, a model (not in a formal sense), that to satisfy a concept of set must consist of

- some objects (the “sets” of the structure);
- some other objects, different from the previous objects (the “urelements” of the structure);
- a binary relation (the membership relation).

Also, sets and the membership relation should be built from the urelements as specified by the iterative concept of set (and this depends on the iterative

[1] In particular, since an ordering is a good order if and only if any property possessed by some element of the ordering field is also owned by the least element, this definition implies the quantification on the properties, as in the answer to question (2) by the proponent of the strong iterative concept.

concept chosen, if the strong one or the weak one). Thus the number (and type) of structures that satisfy a certain concept of set changes (of course) with the change of the concept of set: for example, it will be very difficult for a structure to satisfy the concept of set if the Uniqueness Postulate is true (and therefore it was used to define the concept of set), while it will be considerably easier if the concept of set is founded on structuralism.

If a structure satisfy the strong iterative concept automatically satisfies the weak one, while the reverse is not always true. For example, if we assume that there are no infinite sets and that a finite number of objects always form a set, and also that the Uniqueness Postulate is true, then we will have that pure hereditarily finite sets are the domain of a structure that satisfies the weak concept of pure set but not the strong one. Finally, any structure that satisfies the strong concept of set must satisfy the axioms of ZFC, including the second order versions of the Axiom of Comprehension and Replacement. Instead if it satisfies only the weak concept of set, it has to satisfy at least the Extensionality and the Axiom of Foundation.

4.2 Categoricity results

Now that we have defined what is a structure and how it can satisfy a concept of set we can give the principal of Martin categorical results. To do this, first we have to assume that the concept of *natural number sequence*is categorical (i.e., any two structures $\mathfrak{M}_1$ and $\mathfrak{M}_2$, which consist of objects and with an unary operation, and that satisfy the concept of sequence of numbers, are isomorphic). We must also assume that if $\mathfrak{M}$ is a structure that satisfies the weak concept of set, then it will be possible to assign the sets of $\mathfrak{M}$ to the individual levels. The levels should also be well-ordered (as already mentioned), and this implies that we can use the transfinite induction to prove that any two assignments of levels are isomorphic. Finally, we assume that the Uniqueness Postulate is true.

Given all these assumptions, we can prove the following results:

1. the Uniqueness Postulate implies that at most one structure satisfies the weak concept of set and thus, a fortiori, at most one structure satisfies the strong concept of set;[2]

2. for a finite number of levels s and for a small enough number of levels s', the parts $V_s^{\mathfrak{M}}$ of the structures $\mathfrak{M}$ that satisfy the strong concept of set are isomorphic, and the isomorphism are unique;

3. any two structure that satisfy the concept of set are isomorphic, and the isomorphism are unique.

[2]There is a stronger versione of these theorem, but it is provable only assuming that the concept of sequence of number is categorical and the Uniqueness Postulate.

To prove the first result, we consider two structures $\mathfrak{M}_1$ and $\mathfrak{M}_2$ satisfying the weak concept of sets. If they have the same sets, then the Uniqueness Postulate will also have the same memebership relation, and therefore have the same structure. If they do not have the same sets, they will differ at a level S_n for a set x that belongs only to $\mathfrak{M}_1$. Always for the Uniqueness Postulate, the members of the set x are known, and will belong to a level s_{n-i}. Since $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are distinguished only from the level S_n, members of x will be in $\mathfrak{M}_2$, but this means that in $\mathfrak{M}_2$ a set isn't formed, and then the iterative process stops. Then $\mathfrak{M}_2$ does not satisfy the weak concept of set, but this contradicts our assumptions. Then the two structures are the same also in this case.

Another possible argument is as follows: given the Uniqueness Postulate, all structures that satisfy the weak concept of set can be "merged" in a single structure by taking the union of all their domains. Furthermore, as said before, the individual parts of this union cannot satisfy the weak concept.

As for the second result, we must first give some definitions. If $\mathfrak{M}$ is a structure that satisfies the strong concept of set and if s is a level of $\mathfrak{M}$, denoted by $V^{\mathfrak{M}_s}$ the $\mathfrak{M}$-set whose members are set of $\mathfrak{M}$ before the level s. The strong concept implies that the set $V_s^{\mathfrak{M}}$ is formed precisely at the level s, then there is such a $\mathfrak{M}$-set. The proof of the theorem proceeds by induction to prove that, for all natural numbers n, there exists a unique isomorphism between the sets $V_n^{\mathfrak{M}_1}$ and $V_n^{\mathfrak{M}_2}$. It follows that there exists an isomorphism between $V_\omega^{\mathfrak{M}_1}$ and $V_\omega^{\mathfrak{M}_2}$, and this isomorphism is unique. We can also extend this argument by defining an isomorphism π between $V_{\omega+1}^{\mathfrak{M}_1}$ and $V_{\omega+1}^{\mathfrak{M}_2}$: to show that these two sets are actually only one we must first define a property P_x, which is satisfied if and only if $y = \pi(z)$ for some $z \in \mathfrak{M}_1$ which is a member of an arbitrary $x \in \mathfrak{M}$. If y has the property P_x then y is in $\mathfrak{M}_2$, a member of $V_\omega^{\mathfrak{M}_2}$. Since $\mathfrak{M}_2$ satisfy the strong concept of set, some set of it, of wich the elements have the property P_x, has to be formed at the level ω (or before). For extensionality, this set is unique. Finally, we can define an isomorphism π^* that, with an argument very similar to the previous one, allows us to demonstrate that the sets $V_{\omega+1}^{\mathfrak{M}_1}$ and $V_{\omega+1}^{\mathfrak{M}_2}$ are actually the same set. This process can be repeated indefinitely, demonstrating the existence and uniqueness of the isomorphisms between $V_{\omega+2}^{\mathfrak{M}_1}$ and $V_{\omega+2}^{\mathfrak{M}_2}$, $V_{\omega+3}^{\mathfrak{M}_1}$ and $V_{\omega+3}^{\mathfrak{M}_2}$, and so on. Finally, to prove that the two structures $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are isomorphic, we must use the fact that the levels are well-ordered and that there is an absolutely infinite number of them.

This demonstration, however, has two problematic points: the concept of well-ordering and the use of absolute infinite. In fact, while the use of well ordering under the assumption of the Uniqueness Postulate is less problematic (the property of the individual levels of being well ordered is in fact directly defined, exactly as in the case of the property P_x), the use of the same property in the final part of the proof (when it proves the isomorphism between $\mathfrak{M}_1$ and $\mathfrak{M}_2$) implies quantification on arbitrary functions from the initial segments of one of the structure to the other one, and this cannot be accepted by all.

instead, the only criticism against the use absolute infinite it's just a question of principle: if we do not consider the notion of absolute infinite to be clear enough, is very easy to deny that it is possible to know whether a given collection of objects (in our case, of levels) is an absolute infinite. In our case, however, the only thing we know is that the previous level to that taken into account may not be an absolute infinite, but this comes from the concept of set. Despite this, we can always have doubts on the use of absolute infinity, but we can be prove the same results without using it. To do so, we must assume that the given isomorphism preserves the ordering of the formation of the individual levels and that the functions induced on the levels are surjective. In this way we can prove that the constructed isomorphism has the same properties as the one built with the aid of the absolute infinite.

4.3 Some concluding remarks

The results reported in the previous section are all about uniqueness and categoricity: with them in fact Martin wants to show that the existence of two different structures (and thus, ultimately, two universes) that satisfy the concept of set is not possible. But all of Martin's arguments do not say anything about the existence of this single structure.

As for the strong concept of set, to ensure the existence of a structure that satisfies it we must assume the particularly strong principles. The situation of the weak concept is rather simple: assuming the postulate of uniqueness, affirming the existence of a structure that satisfies the weak concept is not at all problematic (note that the requirements to meet the weak concept are much stricter than those to meet the strong concept together). Sets that belong to this structure will all be those formed in the iterative process.

Parte II

Case study: the Continuum Hypothesis

Capitolo 5

Preliminaries

In these chapters I will discuss the Continuum Hypothesis problem, taking into account, in particular, the pluralistic and anti-pluralistic solutions. As noted in previous chapters, the main battleground between pluralists and not pluralists are the results of independence: all their constructions are nothing more than justifications for these results. Among the results of independence, the first and (probably) the most important remains the one about the Continuum Hypothesis. The importance of the Continuum Hypothesis is due to the fact that its acceptance or not will forge the mathematical universe in totally different manners. For example, a universe in which the Continuum Hypothesis is true would be very similar to the one dreamed by Cantor and similar to the "classical" universe. Instead, in a universe in which the Continuum Hypothesis is false, relatively simple operations, such as cardinal arithmetic, would become considerably more complex.

After a few preliminaries (section 5), where are reported purely mathematical aspect of the Continuum Hypothesis (focusing in particular on the independence results), in section 6 I will discuss the problem of the definiteness of the Continuum Hypothesis. In fact, despite the Continuum Hypothesis was formulated 150 years ago and was already considered a major problem by Hilbert, the fact that it is a well-defined mathematical problem remains to be demonstrated. Obviously, both the pluralists and non-pluralists believe in its definiteness. Finally, in chapter 7 I will consider possible solutions to the problem of the CH.

In questo capitolo riporterò alcuni dei risultati più importanti riguardanti la CH. Nella sezione 5.1 tratterò dei risultati di indipendenza (che come abbiamo più volte detto sono fondamentali). Nella sezione 5.2 invece discuterò di alcune formulazioni alternative della CH.

This chapter will report some of the most important results concerning the CH. In section 5.1 I will discuss the independence results (which, as we have repeatedly said, are essential). Instead section 5.2 will be dedicated to some alternative formulations of the CH.

5.1 The Independence results

The Continuum Hypothesis, since its first formulation by Cantor, not only involves the cardinality of the reals, but it is also of fundamental importance for the understanding of the laws of cardinal arithmetic, and in particular for those of exponentiation. In fact, while the laws that regulate the addition and multiplication of infinite cardinal numbers are trivial, those for the elevation to power are less simple. While to compute addition and multiplication between two infinite cardinals κ and λ we just need the following rule:

$$\kappa + \lambda = \kappa \cdot \lambda = \max \{ \kappa, \lambda \},$$

in the exponentiation case κ^λ the situation is much more interesting.

One of the first results by Cantor was the proof that, for every cardinal κ,

$$2^\kappa > \kappa.$$

Now, this is obvious in the case 2^n, where n is finite. The question that naturally arises is where, within the hierarchy of the alephs, we can place $2^{\aleph_0}$. The importance of this question lies in the fact that $2^{\aleph_0}$ is the cardinality of the continuum, i.e. of the parts of $\mathbb{N}$.

Cantor was the first to formulate an hypothesis regarding this issue: the famous *Continuum Hypothesis* (CH):

Continuum Hypothesis $2^{\aleph_0} = \aleph_1$.

In reality, this is only a special case of the *Generalized Continuum Hypothesis* (GCH):

[Generalized Continuum Hypothesis] For every α, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$.

The main virtue of the GCH is define in a vary elegant manner the power elevation in the case of two infinite cardinals κ and λ. In fact, assuming it, the following rules hold:

- $\kappa \leqslant \lambda \implies \kappa^\lambda = \lambda^+$;
- $\mathrm{cf}(\kappa)^1 \leqslant \lambda \leqslant \kappa \implies \kappa^\lambda = \kappa^+$;
- $\lambda < \mathrm{cf}(\kappa) \implies \kappa^\lambda = \kappa$.

Initially, the only results about the CH and the GCH were the aforementioned Cantor's results, the trivial result that if $\kappa \leqslant \lambda$ then $2^{\kappa\lambda}$ and the König's result that $\mathrm{cf}(2^\kappa) > \kappa$. The reason for this lack of results lies in the independence results. These results argue that we cannot prove niether CH nor GCH from ZFC axioms. First came Gödel's results:

[1]With $\mathrm{cf}(\kappa)$ we mean κ *cofinality*, that is the least cardinality of κ subsets A and B that satisfy the following condition: for each $a \in A$, there exist a $b \in B$ such that $a \leqslant b$.

Theorem 9. *Lets assume that ZFC is consistent. Then, both $ZFC + CH$ and $ZFC+GCHareconsistent$.* To prove that Gödel used the inner models method, that is proves that the CH and the GCH are true in a minimal inner model L of ZFC.

This theorem would not be a problem, since it states that the CH and the GCH are consistent with ZFC axioms. Obviously, the problem was that the two hypothesis are independent from ZFC, although that was not known. In 1938, when Gödel proved this result, there was still hope for a positive resolution of the CH, although its possible independence seemed increasingly likely.

After Gödel's results was Cohen's turn:

Theorem 10. *Lets assume that ZFC is consistent. Then, both $ZFC+\neg CH$ and $ZFC + \neg GCH$ are consistent.*

Similarly to what was done by Gödel, Cohen to prove this theorem invented the *outer models* method. He showed then that the CH and GCH were not true in a *generic extension* $V[G]$ of V. Combining the results of Gödel with those of Cohen we have the proof of the independence of the GCH and the CH from ZFC axioms. In other words, if ZFC is consistent, then we cannot decide, within ZFC, the truth value neither of the CH nor of the GCH.

Finally, the scene was completed by Easton results[2], that proved that the only fact provable inside ZFC are the one already proves by Cantor and König:

Theorem 11. *Lets assume that ZFC is consistent and that F is a definable function on infinite cardinals such that*

- $\kappa \leqslant \lambda \implies F(\kappa) \leqslant F(\lambda)$;
- $F(\kappa) > \kappa$;
- $\mathrm{cf}(F(\kappa)) > \kappa$.

Then $ZFC+$ "For every infinite regular cardinal κ, $2^\kappa = F(\kappa)$" is consistent.

In conclusion, the situation is clear: remaining inside ZFC it is not possible to go beyond these limits of cardinal arithmetic.

5.2 Three alternative versions of the CH

Until now we always referred to the CH in a very general way. In reality, we can formulate three different versions of it:

[2] See [Easton 1963].

- the *interpolant* version;
- the *well ordering* version;
- the *surjective* versione.

These three versions are all equivalent in ZFC, but we can establish a definability constraint to find some interesting differences. In particular, since there is a hierarchy of notions of definability (Borel hierarchy, projective hierarchy and the hierarchy in $L(\mathbb{R})$ and finally the hierarchy of universally Baire sets), we will have that these three versions will be hierarchically ordered, each corresponding to a level of the definability hierarchy.[3]

5.2.1 The interpolant version

The first version we're going to deal with is the interpolant one. This formulation states that there is no *interpolation*, i.e. that there is no infinite set A of real numbers such that the cardinality of A is strictly between that of the natural numbers and the real numbers (in symbols: there is no A such that $|\mathbb{N}| < A < |\mathbb{R}|$). To obtain definable versions of this version we just need to state that there are no definable interpolations, and this leads to an interpolating hierarchy defined accordingly to the notion of definability used. More precisely, for a given class of points Γ in the hierarchy of definable sets of reala, the corresponding interpolant version of the CH states that there is a definable interpolation Γ.

The first result about this version of the CH was the Cantor - Bendixson Theorem, which showed that there was no interpolation in Γ in the case Γ was the class of closed sets of points. This result was improved by Suslin, which showed that this version is true even when Γ is the class of sets Σ^1_1. We cannot prove stronger results while remaining within ZFC: to do this, we have to use much stronger assumptions. In fact we can use the axioms of definable determinacy to show that, if the Δ^1_n-determination holds, then also the interpolating version of the CH holds for the class of points of sets Σ^1_{n+1} (this is a result of Kechris and Martin). If we assume $AD^{L(\mathbb{R})}$, then this version of the CH applies to all sets of reals in $L(\mathbb{R})$. Since, however, both of these assumptions are derived from large cardinal axioms, it is also possible, assuming very strong large cardinals, showto prove even stronger results. In particular, it can be shown that this version of the CH applies to *all* the sets of reals in the definability hierarchy: in fact, if there is a proper class of Woodin cardinals, then this version of the CH applies to all universally Baire sets of reals.

[3]Here the discussion follows [Martin 1976].

5.2.2 La versione del buon ordine

The second formulation states that every well-ordering of the reals has ordering type lower than $\aleph_2$. As before, the corresponding version for a class of points Γ in the hierarchy states that every well ordering (encoded by a set) in Γ has ordering type lower than $\aleph_2$.

Also this version definable determinacy and large cardinals enable us to demonstrate even stronger versions: for example, if $AD^{L(\mathbb{R})}$ holds this version is true for all sets of natural numbers in $L(\mathbb{R})$, while if there is a proper class of Woodin cardinals we can apply it to all universally Baire sets of real.

5.2.3 La versione suriettiva

The third version is probably the one with the most interesting consequences. In fact, as we shall see in section 7.2.1, it allows Woodin to prove, given certain assumptions, the existence of a canonical model of ZFC in which the CH is not valid. This version of the CH states that there is no surjective function $\varrho : \mathbb{R} \mapsto \aleph_2$, or, equivalently, that there is no pre-ordering of $\mathbb{R}$ of length $\aleph_2$. For a class of points Γ in the definable hierarchy, the corresponding version states that there is no function $\varrho : \mathbb{R} \mapsto \aleph_2$ whose encoding is in Γ.

In this case, the axioms of definable determinacy and large cardinals axioms are even more important, as they allow us to set limits on the length of pre-orderings[4]. Let δ^1_n be the supremum of the lengths of pre-orderings of reals Σ^1_n and let $\Theta^{L(\mathbb{R})}$ be the supremum of the pre-orderings of reals in the cases where the pre-ordering is definable, namely in the cases in which belongs to $L(\mathbb{R})$. Starting from the classical result, which showed that $\delta^1_1 = \aleph_1$, the strengthening of this version has been steady. First Martin has shown that $\delta^1_2 \leqslant \aleph_2$ and that $\delta^1_3 \leqslant \aleph_3$; later Kunen and Martin showed that, assuming the Projective Determinacy (PD), it was possible to demonstrate that $\delta^1_4 \leqslant \aleph_4$ (Jackson showed an even stronger version, namely that, for each $n < \omega$, $\delta^1_n \leqslant \aleph_\omega$). At this point we can find a pattern, and imagine an overall result: assuming the existence of infinite Woodin cardinals, this limit is valid, regardless of the size of $2^{\aleph_0}$.

As we will see in the aforementioned section 7.2.1, this version brought to the attempt to prove that these pre-orderings have all shorter lenght than $\aleph_2$ and, more generally, that the large cardinals axiom implies that this version of the CH is valid for all universally Baire sets of reals. Woodin's results, however, disprove these hopes, and even opened the doors to its arguments against the CH.

[4]A pre-ordering is a binary relation that is reflexive and transitive.

Capitolo 6

Is the Continuum Hypothesis a well defined problem?

In this chapter I will discuss the problems regarding the definitness of the CH. In particular, I will discuss Feferman's position, who belivies that the CH is not well defined.[1]

The main point of his position is that the CH is "inherently vague". This means that for Feferman the CH is vague not only in its formulation[2], but it is vague in its core: namely, the concepts of arbitrary set and arbitrary function are undefined, even at the $\mathcal{P}(\mathbb{N})$. Before describing Feferman's argument we have to note that he doesn't deny that the CH is a well formed and defined formula in the language of set theory, but only that, from the philosophical viewpoint, the concepts used and needed to formulate the CH are problematic.

We can define his position as a *conceptual structuralism*: mathematics is just a human construction (thus Feferman is also an anti-platonist), and is about investigating structures. One of the consequences of this definition is that Feferman doesn't accept the notion of *absolutely undecidable* (that is, undecidable relatively to any set of axioms, see [Koellner 2010]). In fact, if a proposition is absolutely undecidable if and only if its mathematical meaning is defined and thus has a fixed truth value.

To prove his thesis, Feferman use three points of argumentation:

- a mental experiment about *The Millennium Prize List* (cfr. [Jaffe 2006]);
- a philosophical argument, based on the definition of conceptual structuralism;

[1] Feferman's arguments can be read in [Feferman 1999] and [Feferman 2000].

[2] In the next chapter we will see that also for Shelah the formulation fo the CH is not the best one, but that doesn't mean that the CH is not defined per se.

- a mathematical argument, in which he propose a logic theory to distinguish between defined and undefined concepts.

The core of his conception is the philosophical argument: the mental experiment is only cited, while the mathematical argument doesn't add any to the philosophical argumentations.

The main point of the mental experiment is that the CH would not be accepted as a millennium problem by the Clay Institute because the notion of truth for usual mathematics (a problem is solved if we can prove that is true or that is false, and not if we can prove its independence)[3], is very different from the notion of truth usually considered by set theorist when studying the CH. Feferman itself doesn't believe much in the strenght of this argument, so that he doesn't consider it enough to decide for the undefiniteness of the CH.

The philosophical argument is a lot more interesting. Feferman believes that present mathematics is dominated by structuralism (abstract algebra, topology, analysis; Bourbaki, category theory). For him mathematical objects do exist only as mental conceptions, although this mental images are based on everyday experience (with operations like counting, order, etc.). This basic conceptions are images of an idealized world made of structures, and not isolated objects. These structures are set of objects connected by relations and easy operations. All of that is known and communicated on a pre-mathematical level, before any axiomatic or logical system is developped. But, while some properties of these structures are explicit and directly derived by the images of the idealized world, some others are implicit, and need some mathematical work to be understand. Thus, all the basic conceptions have a different clearness degree, and on this degree we base our notion of truth. We can always talk about the truth of any conception, but only the conceptions that are totally clear can be considered really true. Mathematical objectivity is based in its stability through repeated communication of its concepts, the study of them and the independent work of a lot of individuals. Thus, mathematical objectivity is a special case of *intersubject objectivity*, that is omnipresent in the *social reality*.

We can find a philosophical justification of this argument in an essay by Searle of 1995, *The Construction of Social Reality*:

> There are portions of the real world, objective facts in the world, that are only facts by human agreement. In a sense there are things that exist only because we believe them to exist. [...] things like money, property, governments, and marriages. Yet many facts regarding these things are 'objective' facts in the sense that they are not a matter of [our] preferences, evaluations, or moral attitudes.

[3] At least, this is the prospective accepted by the Clay Institute when evaluating solutions to millennium problems.

The reality that we assume also for mathematics is the result of intersubject objectivity and not a independent reality in platonic sense. This position doesn't require a total realism regarding the truth values: is completely up to us to decide if a proposition has a defined truth value or not. Feferman prefers to accept realism for the truth values of some proposition, but only in the case of clear and evidents structures (for example, number theory).

How can we apply all of this to the CH? Feferman believes that the continuum is not a unique concept, but various correlated concepts (a geometrical concept, the real line, a set theoretic concept)[4], and that is not clear enough which one of these concepts is the CH referring to. In particular, the only way to define the continuum is to refer to the geometrical continuum or to the real line, but the identity of these two concepts requires an impredicative set theory.

Sets should be defined totalities, determined only by the objects that are in membership relation with them, and independently from how they are defined (if defined). So a set is a defined totality if and only if quantification on it has a determined truth value for every property of elements of the set itself. From this prospective, V is not a defined totality, so quantification without constrain on it is not justified. Thus, Feferman affirms that V is essentialy undefined.

Moreover, for Feferman the assumption of $\mathcal{P}(\mathbb{N})$, $\mathcal{P}(\mathcal{P}(\mathbb{N}))$ can be justified only from a platonist point of view, while for the conceptual structuralist the notion of a totality of arbitrary subsets of any infinite set is undefined (inherently vague).

La conclusione del ragionamento di Feferman consiste nella costruzione di una teoria logica per distinguere i concetti ben definiti da quelli indefiniti. Senza entrare troppo nei dettagli, una proposizione φ è *formalmente definita* nella teoria logica di Feferman se e solo se $\varphi \vee \neg\varphi$ è dimostrabile in esso. In particolare, la CH è sì esprimibile, ma non dimostrabile. Quindi, conclude Feferman, è un concetto essenzialmente vago.

The conclusion of Feferman's argument is the construction of a logic theory to distinguish well defined concepts from undefined one. Without too much details, a proposition φ is *formally defined* in Feferman's logi ctheory is and only if $\varphi \vee \neg\varphi$ is provable in it. In particular, the CH is expressible, but is not provable. Then is a inherently vague concept.

There are some problems in this whole argument: for example, this purely technical conclusion is based on very peculiar assumptions, and it's very probable that changing these assumptions would change radically which concepts are considered defined and which undefined. The main problem for the defenders of CH definiteness is that until an axiom that solve it is found, their position will weaken every day. This is said expicitily by Martin in 1976:

[4] See [Feferman 2009].

> Those who argue that the concept of set is not sufficiently clear to fix the truth-value of CH have a position which is at present difficult to assail. As long as no new axiom is found which decides CH, their case will continue to grow stronger, and our assertion that the meaning of CH is clear will sound more and more empty.

It seems that the only hope to prove the clearness of the CH is to solve the CH.

Capitolo 7

Solutions to the CH

7.1 Le soluzioni pluraliste

This section will discuss the possible solutions of the Hypothesis Continuous proposed by advocates of the realist multiverse. In particular, in section **??** I will focus on Hamkins position, which is especially explicit in dealing with the problem and, moreover, is the only one (with respect to the positions of Steel, Woodin and Friedman) evidently supports the existence of multiple universes (as we saw the other positions on this are much more ambiguous). Concludes a discussion of Shelah's results regarding the GCH (section 7.1.2).

7.1.1 Hamkins' solution to the CH

The starting point of Hamkins' argumentations is the following theorem:

Theorem 12. *The universe V has the following extensions through forcing:*

- $V[G]$. *in which no cardinal collapses, such that* $V[G] \models CH$*;*
- $V[H]$*, in which no new reals are added, such that* $V[H] \models \neg CH$*.*

This means that both the CH and $\neg CH$ are forcible on any model of set theory, and this implies that each model is very similar to models in which the CH is forced in the "opposite" way. In other words, we can distinguish many models exclusively by the answer given to the CH.

For Hamkins, because of this fact, the CH is no longer an open question: its solution is all the knowledge on it that the set theorists have accumulated over the years through the use of forcing. In fact, for Hamkins, we have now come to a deep understanding of how and why the CH fails or is satisfied in a given model of set theory, and this knowledge turns out to be the "answer" to the problem of the Continuum Hypothesis.

Hamkins' main argument is the refutation of what he call the "ideal solution". This is divided in two steps:

1. Build a statement Φ that express a principle of set theory *clearly true*;

2. Prove that Φ determines the CH, that is, prove that $\Phi \implies CH$ or that $\Phi \implies \neg CH$.

The chosen principle Φ must be "clearly true" like the other axioms of set theory: it must therefore express a principle whose truth is unanimously shared. Such a solution is " perfect " as solves the CH leaving no chance to reply: everyone has to accept Φ and its consequences.

For Hamkins this solution is now impossible and unfeasible, because of all the experience we have accumulated seeing worlds in which CH holds and others where instead $\neg CH$ holds. In fact, our current situation is not limited to the results of independence, but actually exceeds the fact that CH cannot be demonstrated from ZFC. Instead, we know in detail how the worlds, where the CH is true or false, are built and we know how to build these models and how, starting either from a model in which the CH is true or a model in which instead it is false. So for Hamkins, *any* ideal solution to the CH would be unsatisfactory. This holds both in the case in which is proved $\Phi \implies CH$ and in the case in which is proved $\Phi \implies \neg CH$. In fact, for Hamkin's, either solution would mean that we lose all those worlds in which the opposite case is instead true, and where we "lived" for all these years. This then leads Hamkins to the rejection of step (2) of the ideal solution.

A possible example of ideal solution is the Axiom of Symmetry proposed by Freiling in [Freiling 1986]. The goal of this article is to *philosophically* prove the falsity of the CH. Freiling's attempt is a good example of what Hamkins denote as ideal solution: despite Freiling proposes an axiom and from that axiom proceeds to prove the negation of the CH, its demonstration has never been accepted as a proof of the falsity of CH.

The Axiom of Symmetry proposed by Freiling states that for any function f from the reals to countable sets of real, there must be two reals x and y such that $y \notin f(x)$ and $x \notin f(y)$. At this point, lets choose a first x. Since the set $f(x)$ is countable, for Freiling is highly likely that the second real y we choose will not belong to $f(x)$. Moreover, as the order in which these choices are made does not matter, we can conclude by symmetry that $x \notin f(y)$. So we have some intuitive reason to believe not only in the existence of a pair (x, y) with this property, but also that all possible pairs have this property! To better understand the following example may be helpful. Assuming you want to throw two darts at a target: the first will hit the point x, in section 57 points (which is our $f(x)$). Intuitively, we consider very improbable (if not, in some cases, even physically impossible) that the second dart will land at the same spot or in the same section, while the fact that is going to hit a point y in another section (i.e. $f(y)$) is considered "normal". The axiom is therefore compatible with our pre-mathematical intuitions, and thus fulfills step (1) of the ideal solution. Freiling at this point moves to the demonstration (in ZFC) that the Axiom of Symmetry is equivalent to $\neg CH$, so demonstrating that $AS \iff \neg CH$[1].

This demonstration, however, is not accepted as a proof of the falsity of CH. The main objection against Freiling's proof is its implicit assumption, for a given function f, that several sets were measured, in particular the set $\{ (x, y) \mid y \in f(x) \}$. Freiling's response focuses on the fact that the Axiom of Symmetry is based on pre-mathematical insights, and then forego any development of measure theory. His arguments are therefore at the same intuitive level than those who try to justify the basic properties of measure. The problem is that, having now a deep understanding of measure theory, as well as of non-measurable functions and sets, intuitive arguments and pre-mathematicians intuitions like Freiling's one are totally inadequate. Why prefer a pre-mathematical argument to decades of mathematical practice on the same topic? Of course, we can change Freiling's to make it more acceptable, but it would remain unacceptable for the same reasons as the original: our experience with "strangely behaving" functions and sets it is now too much. Ironically, it is much more common inverting Freling's argument to prove the inevitable difficulties that arise if we not accept the CH.

The situation is much more clear if we make another example. We call the next principle *Power Set Size Axiom* (PSA):

$$\forall x, y[|x| < |y| \implies |\mathcal{P}(x)| < |\mathcal{P}(y)|].$$

Intuitively, this principle states something obvious: strictly larger set have strictly more subsets. A principle so obvious, that outside the logical and set theoretical community this principle is considered true exactly as we consider it any other basic set theory axiom.

Instead if we interrogate a logician or a set theorist, the answer will be totally different. First, the axiom PSA is independent from ZFC. Moreover, it is possible to construct ZFC models with counterexamples of this principle: for example, it is possible to use Easton Theorem to build very atypical patterns for the function $\kappa \mapsto 2^\kappa$, and even Cohen's original model $ZFC + \neg CH$ has $2^\omega = 2^{\omega_1}$ (this isLuzin Hypothesis, see [Luzin 1935]). In addition, Martin's axiom implies $2^\omega = 2^\kappa$, and this can lead to even more serious violations of the PSA when the CH is not true. So not only the set theorist knows that the axiom PSA can be not satisfied, but also has experience of models where PSA is actually not satisfied (for example, is incompatible with Martin Maximum MM, or with the forcing axiom PFA). The fact that this axiom is not accepted by set theorists amongst the basic axioms perfectly exemplifies Hamkins's argument. In fact the axiom PSA expresses a basic idea about set size, it is intuitively true, and it is also consistent with

[1] The technical details of the demonstration are found in [Freiling 1986] while in [Hamkins 2012 we can find a detailed discussion.

the other axioms of ZFC (as well as being a consequence of GCH). Unlike in the case of Freiling's Symmetry Axiom, the axiom PSA seems to have all what it takes to be "officially" accepted. But it is rejected because we have too much experience of how the principle can be violated, and we have no plans to stop working with models in which it is violated.

In conclusion, Hamkins does not eliminate the possibility of finding a proposition Φ that decides the problem of the CH. On the contrary, Hamkins's argument is more general: if it were a proposition like that, the proof would not be accepted, because we do not want to abandon decades of research and knowledge about the opposite situation (for example, if it proves that the CH is true, it would be impossible for Hamkins to give up all models in which instead it is false). It should be noted Hamkins's argument can also be used in the case of non-ideal solutions, such as that of Woodin based on the Ω-logic. In fact, the core of Hamkins's argument, that any proof that decides the CH would make illusory the experience gained so far, can be applied in exactly the same way.

7.1.2 Shelah's position

Shelah's solution[2] tt is perhaps the most characteristic. In fact, unlike other proposals, which in any case never try to attack the problem openly, Shelah try to solve the Generalized Continuum Hypothesis directly. The starting point of Shelah is the solution to Hilbert's Fifth Problem[3]: the peculiarity of this problem is that, as formulated by Hilbert, was false. Andrew Gleason, which contributed greatly to its resolution, proposed to totally change the formulation. Shelah offers exactly the same thing with regard to the problem of the CH.

The general idea is that the CH does not regard the reals number, but the laws of cardinal arithmetic. His reformulation of the GCH then tries to bring this fact to the fore. First of all, Shelah reformulates the GCH (which in its original formulation is very simple: $2^{\aleph_\alpha} = \aleph_{\alpha+1}$) in the following equivalent way: given two regular cardinals $\kappa < \lambda$, $\lambda^\kappa = \lambda$. At this point, and also taking into account the independence results, Shelah proposes to reformulate the GCH as follows: "for each regular cardinal λ and for every regular cardinal $\kappa < \beth_\omega$ large enough such that $\kappa < \lambda$, then λ raised to the *revisited power* κ is equal to λ". The revised power κ for Shelah is the least cardinality of the subsets family of λ where each one has cardinality κ such that every other subset of λ of cardinality κ is included in the union of a number strictly less than κ of family members.

In other words, for Shelah the real issue is not how many points there are on the real line, but how many "small" subsets of a given set are needed to cover each small subset with only some of them. And, surprisingly, Shelah

[2]See [Shelah 2000].

[3]The fifth problem was about Lie groups characterization

was able to show, through applications of his *pcf*, the existence of a limit to the exponential function. In particular, this limit is $2^{\aleph_\omega} \leqslant 2^{\aleph_0} + \aleph_{\aleph_{\omega_4}}$. Of course, this result is very different from the CH, because, as we have seen, Shelah changes the question radically (and also deals with much larger sets). However, despite this, its results change the course of the last fifty years of studies, until now filled mostly with independence results.

7.2 Woodin's solution: $\neg CH$

In questa sezione tratterò dei risultati di Woodin per la falsità di CH. Nella sezione 7.2.1 riporterò i risultati di Woodin che hanno di fatto eliminato ogni possibilità per il successo del programma di Foreman - Magidor. Questi risultati hanno permesso a Woodin di identificare un modello "canonico" in cui la CH fosse falsa (sezione 7.2.2).

This section will discuss the Woodin's results for the falsity of the CH. In the section 7.2.1 I will report the Woodin's results that have effectively eliminated any chance of success for the Foreman - Magidor program. These results have allowed Woodin to identify a "canonical" model in which the CH is false (section **??**).

7.2.1 The Foreman - Magidor problem

The Foreman and Magidor's goal[4] was showing that the large cardinals axioms implied that the pre-ordering Θ on real numbers in the constructive universe L was less than $\aleph_2$ and, more generally, that $\Theta^{L(A,\mathbb{R})} \leqslant \aleph_2$ for each universally Baire set A. In other words, this meant proving that the large cardinals axioms implied that the surjective version of the CH was true for all sets in $L(\mathbb{R})$ and, more generally, for all universally Baire sets.

This program was based on Shelah, Foreman and Magidor's results on Martin Maximum (MM)[5], that showed that assuming a large cardinal axiom was always possible to apply forcing to obtain a steep ideal on $\aleph_2$ without collapsing it. The strategy adopted is divided into two distinct steps:

1. strengthen the Shelah, Foreman and Magidor's results showing that, assuming a large cardinal axiom, it was always possible to use forcing to obtain a set of cardinality $\aleph_2$ that satisies a reasonable number of complete types, without collapsing $\aleph_2$ to a lower cardinality (i.e., we can get a saturated ideal on $\aleph_2$);

2. proving that the existence of such a set implies $\Theta^{L(\mathbb{R})} \leqslant \aleph_2$ and, more generally, $\Theta^{L(A,\mathbb{R})} \leqslant \aleph_2$ for every universally Baire set A.

[4]For more information on their program, see [Foreman&Magidor 1995] and [Woodin 1999].

[5]See [Foreman, Magidor&Shelah 1988].

This would prove that $\Theta^{L(\mathbb{R})} \leqslant \aleph_2$ and, more generally, that $\Theta^{L(A,\mathbb{R})} \leqslant \aleph_2$. To do this, we assume large cardinal axioms to the level required in (1) and (2) and the existence of a proper class of Woodin cardinals. Suppose also, by contradiction, that there is a pre-ordering in $L(\mathbb{R})$ of length $\aleph_2$. Now, using (1) to force a saturated ideal on $\aleph_2$ without letting it collapse, in this extension by forcing the original pre-ordering will still be a pre-ordering for $L(\mathbb{R})$ of length $\aleph_2$. This, however, contradicts (2), then the large cardinal axiom previously assumed implies that $\Theta^{L(\mathbb{R})} \leqslant \aleph_2$. We can apply this same reasoning also to the more general case, to prove that $\Theta^{L(A,\mathbb{R})} \leqslant \aleph_2$.

All of this was made totally useless by the following theorem proved by Woodin:

Theorem 13. *Lets assume that a set of cardinality $\aleph_1$ satisfies the highest number of complete types and that there exists a measurable cardinal. Then, $\delta^1_2 = \aleph_2$.*

The main problem for the Foreman-Magidor program is that the hypothesis of this theorem can always be forced assuming large cardinals, so is possible to prove $\Theta^{L(\mathbb{R})} > \aleph_2$ (indeed was proved that $\delta^1_3 > \aleph_2$), the exact opposite result!

The problematic problem it is not the second step: in fact, while Foreman and Magidor had only an approximation of that fact, Woodin manage to prove its truth:

Theorem 14. *Lets assume that there exists a proper class of Woodin cardinals and that there exists a saturated ideal on $\aleph_2$. Then for every $A \in \Gamma^\infty$ holds $\Theta^{L(A,\mathbb{R})} \leqslant \aleph_2$.*

So the problem of Foreman and Magidor's argument was in the first step.

However the core of this argument is another: assuming large cardinals axiom ($AD^{L(\mathbb{R})}$ is enough), although it is known how to produce outer models for which is $\delta^1_3 > \aleph_2$ holds, it is not known how to produce them for $\delta^1_3 > \aleph_3$, or for $\Theta^{L(\mathbb{R})} > \aleph_3$. We can therefore speculate that, assuming $ZFC + AD^{L(\mathbb{R})}$, it is possible to demonstrate $\Theta^{L(\mathbb{R})} \leqslant \aleph_3$. Despite this result does not imply that the large cardinals can eliminate the possibility that the CH is false, they can eliminate the possibility that $2^{\aleph_0} = \aleph_2$ is true.

7.2.2 $\mathbb{P}_{\max}$ e $\neg CH$

Woodin's results of the previous section led Woodin himself to look for a canonical model in which the CH was false and, that is such that his theory cannot be altered by forcing on the sets in the presence of large cardinals. Woodin mainly based his argument on two facts: first, we know that in the presence of large cardinals the second order arithmetic theory and the theory of $L(\mathbb{R})$ are invariant with respect to forcing (and this shows that our independence techniques cannot be used to establish the independence of second order arithmetical issues or $L(\mathbb{R})$ issues if there are large cardinals). Secondly, experience has shown us that the large cardinals axioms seem to respond to all the independence issues of second order arithmetic and to $L(\mathbb{R})$ issue. Moreover, the forcing invariance theorems invariance show that these axioms are actually "complete".

From this it follows that if $\mathbb{P}$ is any homogeneous partial ordering in $L(\mathbb{R})$ then the generic extension $L(\mathbb{R})^{\mathbb{P}}$ inherits the generic absoluteness $L(\mathbb{R})$. In other words, this means that given a proposition φ in $L(\mathbb{R})$, this will not change truth value moving to $L(\mathbb{R})^{\mathbb{P}}$. The main result was discovering the existence of a special partial ordering $\mathbb{P}_{\max}$ that has this property. In addition, the model $L(\mathbb{R})^{\mathbb{P}_{\max}}$ satisfies $ZFC + \neg CH$. The main feature of this model is that it is "maximal" compared to sentences of a certain complexity, the consistency of which can be demonstrated by forcing on the model. This means that if these statements *can* be true, then *should* be true in the model.

Without going into too much technical detail (present in [Woodin 1999]), we can say that Woodin's results essentially show two points: first, that the theory $L(\mathbb{R})^{\mathbb{P}_{\max}}$ is "actually complete", that is it i invariant with respect to forcing. Furthermore, the model $L(\mathbb{R})^{\mathbb{P}_{\max}}$ is "maximal", that is it satisfies all Π_2-propositions that can be satisfied.

At this point Woodin proposes his solution to the CH. In fact, defining the following axiom

Axiom 7 (Axiom $(*)$). *$AD^{L(\mathbb{R})}$ is satisfied and $L(P(\omega_1))$ is an extention$\mathbb{P}_{\max}$-generic of $L(\mathbb{R})$.*

we can prove the next theorem

Theorem 15. *Lets assume $(*)$. Then $2^{\omega} = \aleph_2$.*

We can summarize these results in Ω-logica. In fact, assuming the Strong Ω Conjecture[6], we can prove the existence of a theory of $H(\omega_2)$ that implies $\neg CH$. Moreover, in this theory is satisfied $2^{\aleph_0} = \aleph_2$.[7]

[6] The details are very technical, can be found in [Woodin 1999].

[7] For more details, see [Woodin 2001a], [Woodin 2001b], [Woodin 2005a] and [Woodin 2005b].

Capitolo 8

Concluding remarks

We have already seen that both multiverse and the single universe have some problems. In fact, any conceptions of the multiverse will have problems defining the notion of truth. In Hamkins case we accept any universe without any criterion. Steel's position is more reasonable, but right now feels still incomplete, since the technical results supporting it are few (although the axiom $V = HOD$ is a good start). Lastly, Woodin's position is too peculiar to be taken into consideration: indeed his assuptions are so strong that they are very diffucult to be accepted without any discussion. For example, considering the forcing as invariant through the whole multiverse is a total overturn of its premises: finding atypical models, in which some propositions have different truth value compared to the standard. Among all the multiverse positions the most useful one is probably Shelah's formalism: in fact, without having to engage in ontological and platonic assumptions, Shelah can have all the philosophical and technical advantages of the multiverse.

Yet, also the more simple conception of one single universe is full of difficulties. The main one, Martin's arguments to proves its existence (categoricity), are still very vague. As in Woodin case, the assuptions are very strong and not always "plain and clear". Moreover avoiding to assume the Uniqueness Postulate, though understandable from Martin's point of view, is very difficult to justify (it's the clearest of all the assumptions). Also, in the case of a realist monism the difficult of justifying the existence of other models (universes) than the one "true and real" would be a real problem.

This account is not simplyfied by the solutions to the CH. Woodin's solution is flawed by the same problems of his general conception of the multiverse: asssumptions that are too strong to be easily accepted. This undermine his conclusion against the CH. In fact, although his proof is technical unexceptionable, is strill based on his assumptions, in particular the Ω Conjecture. But just having a different conception of the notion of truth is sufficient to weaken the tenability of the Ω Conjecture, and so the whole proof is undermined. At the same level is Hamkins' conception:

it's just the acceptance of the status quo, without showing any possible progression of research. In regards to these problems connected to the CH is still Shelah that showed how to move forward our knowledge. In fact we have already seen how Shelah proved a result that is not limitative about the GCH, adding a lot to the field. Moreover, Shelah's results are based on a very simply machinary, already present before Cohen's work (and they do not require any strong and unusual assumption).[1]

Currently there is a research project, proposed by Maddy[2], that is very peculiar and interesting. Since Maddy's thought is very similar to Gödel's, her program is only about ZFC and can be considered a realist. Her objective is to determined the truth or the plausibility of some axioms that are indipendent from ZFC. Moreover, these axioms are candidates to extend ZFC (e.g., two of this axioms are $V = L$ and GCH). To do this, Maddy defines "plausibility" in two part: "MAXIMIZE" and "UNIFY". The first part (MAXIMIZE) is about the "power" of the axiom took into consideration. In other words, every axiom that we assume has to be the most powerful possible, that is must be capable of proving the most results possible, without putting any limitation other than consisteny. The secon part (UNIFY) regards the "foundationality" of the theory: the final objective is a single system in which every structure and every object of mathematics can be modelled. The axioms in the intersection of these two definition are defined plausible by Maddy. For example, $V = L$ is not plausible by this definition, while the GCH it is.

In conclusion, what could be a good solution to the CH? The simpliest one is to assume it as an axiom in the theory we are taking into consideration (for example, taking $ZFC + CH$). Otherwise, is possibile to follow the classic road and search for an axiom that would imply the CH. From this point of view the axiom proposed by Steel, $V = HOD$, is very promising.

[1] All the machinary used by Shelah can be seen already in [Sierpinski 1935].

[2] In [Maddy 1997].