Week 8

Solutions to exercises on first order models

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1. Prove the following (tip: try first to prove $\neg(\neg\varphi \wedge \neg\psi) \vdash_{CPL} (\varphi \vee \psi)$. To do so, you will need to prove two different contradiction in two different branches, and then a final third contradiction connecting all the branches):

$$\vdash_{CPL} \neg(\neg\varphi \wedge \neg\psi) \to (\varphi \vee \psi)$$

   - On of the possible proofs is the following:

$$\dfrac{\dfrac{[\neg(\neg\varphi \wedge \psi)]^1 \qquad \dfrac{\dfrac{\dfrac{[\neg(\varphi \vee \psi)]^2 \quad \dfrac{[\varphi]^3}{\varphi \vee \psi}\langle \mathrm{I}\vee\rangle}{\bot}\langle \mathrm{E}\neg\rangle}{\neg\varphi}\langle \mathrm{I}\neg\rangle,3 \qquad \dfrac{\dfrac{[\neg(\varphi \vee \psi)]^2 \quad \dfrac{[\psi]^4}{\varphi \vee \psi}\langle \mathrm{I}\vee\rangle}{\bot}\langle \mathrm{E}\neg\rangle}{\neg\psi}\langle \mathrm{I}\neg\rangle,4}{\neg\varphi \wedge \neg\psi}\langle \mathrm{I}\wedge\rangle}{\dfrac{\bot}{\varphi \vee \psi}\langle \mathrm{CR}\rangle,2}\langle \mathrm{E}\neg\rangle}{\neg(\neg\varphi \wedge \neg\psi) \to (\varphi \vee \psi)}\langle \mathrm{I}\to\rangle,1$$

2. Check if the models given at the end of the notes satisfies the other two sub-formulas of the formula given:

   a) $\mathcal{M} \models \forall y[A(y,x)]$;

      - To solve this exercise, you need to use the model $\mathcal{M}$ given at the end of the notes, i.e. the one with domain $\mathcal{D}^{\mathcal{M}} = \{1,2,3\}$, the constant interpretation $c^{\mathcal{M}} = 3$ and relation interpretation $A^{\mathcal{M}}(x,y) = x > y$.
      - For the first formula you need to substitute to the y all the elements of the domain, since it is in the scope of a universal quantifier. However, the x is *free*, i.e. without any quantifier, so you can choose whatever you want (usually you pick something that would satisfy/not satisfy the model depending on your needs).
      - In this case does not matter, since whatever we choose for x we end up with one case in which $y > x$ is false.
      - Thus, the model $\mathcal{M}$ does not satisfies this formula, in symbols $\mathcal{M} \not\models \forall y[A(y,x)]$.

   b) $\mathcal{M} \models \forall y[A(z,x)]$.

      - For this second formula you need to do the same, but notice that *both* variables are free. Just pick a case in which $z > x$ is true, and you have proved that the model satisfies it.

3. Build another model $\mathcal{N}$ that does not satisfy the formulas above BUT satisfies the other sub-formula:

   a) $\mathcal{N} \not\models \forall y[A(y,x)]$;

   b) $\mathcal{N} \not\models \forall y[A(z,x)]$;

c) $\mathcal{N} \models \exists x[A(z, c)]$.

- The process to solve this exercise is just the same: invent a model (pick a domain, an interpretation for the constants and for the relations) and check it.
- In this case, the following model does the trick:
  - $\mathcal{D}^{\mathcal{N}} = \{2, 4, 6\}$;
  - $c^{\mathcal{N}} = 6$;
  - $A^{\mathcal{N}}(x, y) = x < y$