Week 3

Explanation of the solutions to the exercises on truth tables

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Since this week exercises were a little bit more difficult than the others, I will now give you the solutions and explain something about them. Hopefully, this will make everything clearer.

1. Choose one of the pair of connectives $(\neg, \vee)$, $(\neg, \rightarrow)$, $(\neg, \leftrightarrow)$ and define all the other connectives with it.

Let's start with the first exercise. This one asks you to find an equivalent formulation of the logical connectives $(\neg, \vee, \wedge, \rightarrow, \leftrightarrow)$, using only two of them ($\neg$ plus one of the other). An equivalent formulation means writing a formula using the least amount of complexity (so nothing long 3 lines), only two of the logical connectives, such that the truth tables of the two formulas (the new one and the original one of the connective) are the same. This can be done for any combination of negation plus another connective, i.e. for $(\neg, \vee)$, $(\neg, \rightarrow)$, $(\neg, \leftrightarrow)$.

For example, Let's decide we want to write the equivalent formulations using $\neg, \vee$. We need to find a formulation for $\rightarrow, \wedge$ and $\leftrightarrow$, since obviously $\neg$ and $\vee$ are just the same. We start with $\wedge$:

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

We want to write a formula, using only $\neg, \vee$, with the same truth table for the final column. In this case, we note that we can "translate" the truth table in natural language, and say that "a conjunction is true if and only if both conjuncts are true, otherwise is false". Sadly we cannot directly translate this sentence in a formula with $\neg, \vee$, but we can try to explicit that "otherwise". So we can say that "a conjunction is false if and only if p is false, or q is false, or both p and q are false". As you can see, we are now on the right direction: we have a sentence that uses the disjunction and it describes the truth table of conjunction. We can go further: "it is not the case for a conjunction, that p is false or q is false". We can now write this in symbolic form:

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$\neg(\neg p \vee \neg q)$
1	1	0	0	0	1
1	0	0	1	1	0
0	1	1	0	1	0
0	0	1	1	1	0

The process is the same for all the other cases and connectives: try to describe the final column in the terms of the connectives you want to use, and then try to write from that a formula and check its truth table.

Another way to approach this is trying to write a tautology with the two formulas (the original connective and the equivalent formulation) as the two parts of a biconditional. Since the biconditional is true if and only if both parts have the same truth value, this also yields the right solution. So, in our example, we would claim that the formula

$$(p \wedge q) \leftrightarrow (\neg(\neg p \vee \neg q))$$

is a tautology. If you check the truth tables, you can find out that this is indeed the case:

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$\neg(\neg p \vee \neg q)$	$p \wedge q$	$(p \wedge q) \leftrightarrow (\neg(\neg p \vee \neg q))$
1	1	0	0	0	1	1	1
1	0	0	1	1	0	0	1
0	1	1	0	1	0	0	1
0	0	1	1	1	0	0	1

2. The following connective is called "Sheffer Stroke", and can be translated as "not both": $p|q$ means "not both p and q. The behaviour of this connective is explained in this truth table:

p	q	$p\|q$
1	1	0
1	0	1
0	1	1
0	0	1

   This connective is special, since with it is possible to define *all other logical connectives*. Try to do it. *Hint*: Try to keep it as simple as possible: for the negation just one Sheffer Stroke is enough, for implication you need two, for conjunction and disjunction 3 and for biconditional 5.

The second exercise is just the same, but with the new connective described in the exercise itself, i.e. the Sheffer Stroke. So you are trying to find the second half of the following formulas:

1. $(\neg p) \leftrightarrow ?$;
2. $(p \wedge q) \leftrightarrow ?$;
3. $(p \vee q) \leftrightarrow ?$;
4. $(p \rightarrow q) \leftrightarrow ?$
5. $(p \leftrightarrow q) \leftrightarrow ?$

The process is just the same as the first exercise: try to translate the truth table last column in such a way that uses the new connective (in this case, "not both"), and then try to translate that in a formula. Moreover, look at the hint! There are only a finite number of possibilities you have to check, even if you have completely no clue of what to do (admittedly, this is more time consuming).

For example, Let's consider negation. We know from the hint that we only need a Sheffer Stroke, so we are trying to find the following:

$$\neg p \leftrightarrow ?|?$$

To find the right solution, you need to check only two combination: $p|p$ and $p|q$.

Knowing how the negation works, everything else follows more easily, since all the truth tables can translated in the form "[connective] is true/false if and only if *not both* the following cases". For example, the disjunction can be formulated as "it is not the case that both disjunctions are false". Or, more closely to the Sheffer Stroke, "it is not the case that both p and q are false", thus "not both p is false and q is false". This can be then translated in a formula, since we already know how to make a negation:

p	q	$p\|p$	$q\|q$	$(p\|p)\|(q\|q)$
1	1	0	0	1
1	0	0	1	1
0	1	1	0	1
0	0	1	1	0