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The V-logic Multiverse

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Structure of the Presentation

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Preliminaries

- The authors are currently working on a research project bearing the same title (therefore, feedback from audience is especially welcome!)
- Most current work on the V-logic multiverse springs from/expands on previous work conducted within the Hyperuniverse Programme (among others, [Antos et al., 2015], [Friedman, 2016], [Barton and Friedman, 2017], [Antos et al., nd])
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The Multiverse: Two Strategies

Compare the following two main strategies:

Modelism

The ZFC axioms (or any other theory of sets T, for that matter) are *incomplete*. How do we know that? Through 'building' the *models* of ZFC (of T). *Ergo*, in the *metatheory* of ZFC (of T), we may argue about (and study) *the multiverse* of set theory.

Foundational Multiversism

Universes of set theory are a special kind of *objects*. The main task of a multiverse theory is that of providing an account not only of *sets*, but also of *universes* (which means that our theory should be purposefully designed to also incorporate a description of *universes*).

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A Useful Heuristic: Väänänen's Multiverse Concept

Optimality of ZFC

The concept of set is *sufficiently determinate* to generate the structure $(V, \in)$, and a collection of axioms (ZFC) which 'describes' it.

Moreover, all properties of sets not *uniquely* spelt out by ZFC (by the concept of set) 'co-exist in' V ([Väänänen, 2014]).

Thus, one could say that V inherits the *indeterminacy* of the concept of set as far as 'truths beyond ZFC' are concerned.

Let $\mathbb{V}_{mult}$ be the collection of all V's such that each of them satisfies ZFC and each one differs from another 'at the edges'.

The purpose of our multiverse theory is precisely to describe $\mathbb{V}_{mult}$.

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The Hyperuniverse Programme (HP)

- HP[1] manages to vindicate $\mathbb{V}_{mult}$ by assuming that:
  1. V is countable.
  2. *Width extensions* of V can be dealt with by 'theories' in a structure 'built around' V (see next slides).[2]

The Challenge

Assume V is uncountable. Our project aims to:

1. Keep the *definability* of 'width extensions' of V.
2. Assert the *existence* of a wide variety of 'universes'.

[1] See [Antos et al., 2015], [Friedman, 2016], [Barton and Friedman, 2017], [Antos et al., nd] for details.

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Constraints (for a Theory of the Width Multiverse)

Constraint 1

Given V, and a (width) extension W of V, V and W should be 'standard' in our theory (unwanted interpretations should be ruled out).

Constraint 2

Whenever we have, by 'standard' reasoning, that $W \models \varphi$, for some $W \models T$, where W is an outer model of V and T is our 'base theory', then our axioms should be able to state that W is a member of the multiverse.

Constraint 3 (Completeness)

$T \models \varphi \implies T \vdash \varphi$ (the logic which captures the axioms should be complete).

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Infinitary Logics

Let $\mathcal{L}_{\kappa,\lambda}$ be an infinitary language (with $\lambda < \kappa$), allowing the formation of:

1. conjunctions and disjunctions of length $< \kappa$
2. quantification over $< \lambda$ variables

Fact

Infinitary logics have a stronger *expressive power* than first-order logic. The use of one of such logics will ensure that Constraint 1 is met: the representation of 'width extensions of V' will rule out 'unwanted' interpretations.

Consider an example of $\mathcal{L}_{\omega_1,\omega}$: in ω-logic, all models of arithmetic are *isomorphic* to the 'standard model'.

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- a special constant $\bar{W}$, denoting a generic outer model of the ground universe
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Proofs in V-logic: Admissible Sets

We know that proofs may be coded by sets. In V-logic, proofs are coded by sets in $Hyp(V)$, which is the *least admissible set* after V.

Admissible Set [Barwise, 1975]

An admissible set over $\mathfrak{M}$ is a model $\mathfrak{A}_{\mathfrak{M}}$ of *KPU* of the form $\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A, \in, ...)$. A *pure* admissible set over $\mathfrak{M}$ is an admissible set, and $\mathfrak{M}$ does not have urelements (a set $\mathbb{A}$ s.t. $KP \models \mathbb{A}$).

Least Admissible Set

The smallest admissible set over $\mathfrak{M}$ (denoted $Hyp_{\mathfrak{M}}$) is the *intersection* of *all* admissibles over $\mathfrak{M}$ (and is equivalent to L_α, the α-th stage of the constructible universe, where α is the *least admissible ordinal* over $\mathfrak{M}$).

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World Existence and $Hyp(V)$

Therefore, in V-logic, $Hyp(V)$ (henceforth, V^+) is just some $L_\alpha(V)$. Codes of proofs in V-logic are in V^+.

Now, suppose we want to assert that there exists a 'universe' W, a width extension of V.

We proceed *syntactically*: the existence of such a world is equivalent to the *proof* of the following consistency statement:

$$Con(T + \varphi)$$

where T is our base theory (BST), $\varphi =$ "$\bar{W} \models \psi$", and ψ is some property of $\bar{W}$.

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Proofs and Universes

Claim (V-logic)

For each world W extending V and defining property ψ, we have a proof code of $\varphi = Con(T + \psi)$ in V^+.

The property ψ may be chosen in such a way as to express some relevant feature of the model in question.

(for instance, for W a *set-generic extension* of the ground universe, we may characterise W as 'containing a $\mathbb{P}$-generic filter G over V and satisfy ψ').

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Width Extensions

By using the mentioned coding, we may produce universes of all ‘relevant’ kinds, that is, all ‘relevant’ *width extensions* of V.

In particular, we may have:

1. Set-Generic Extensions (‘W is s.t. W contains a $\mathbb{P}$-generic G over V and satisfies ψ’)
2. Class-Generic Extensions (as above, with some modifications)
3. Hyperclass-Generic Extensions (ditto)
4. All kinds of forcing extensions of V
5. *Inner models* of all models defined in 1–4

Thus, Constraint 2 will also be met: models of all ‘relevant’ kinds will belong to the (width) multiverse.

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Summary of Syntactic Multiverse Generation

- In V-logic we have: if $BST + \varphi$ (where BST is our base theory) is consistent, then there exists an outer model W of V such that $W \models \psi$.
- Informally, the multiverse may be seen as a tree: at the root we have the BST chosen, and at every node, a $Con(BST + \varphi)$ statement, where φ asserts that ψ is some *further fragment* of set-theoretic truth
- **A word of caution**: at this stage, we're not assuming that W really 'exists'; only that it can be dealt with by a theory T in V^+

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- In V-logic we have: if $BST + \varphi$ (where BST is our base theory) is consistent, then there exists an outer model W of V such that $W \models \psi$.
- Informally, the multiverse may be seen as a tree: at the root we have the BST chosen, and at every node, a $Con(BST + \varphi)$ statement, where φ asserts that ψ is some *further fragment* of set-theoretic truth
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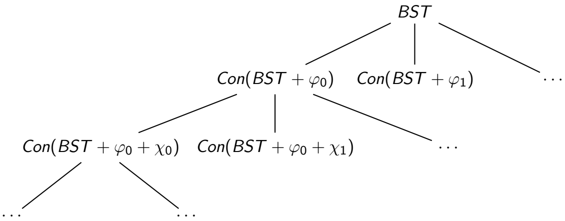
The 'Multiverse Tree'



Structure of the Presentation

Deductive Apparatus: Rules

Modus ponens If $\Gamma \vdash_V \varphi$ and $\Gamma \vdash_V (\varphi \rightarrow \psi)$ then $\Gamma \vdash_V \psi$.

Generalisation If $\Gamma \vdash_V (\varphi \rightarrow \psi(v_n))$ and v_n is bounded in φ then $\Gamma \vdash_V (\varphi \rightarrow \forall v_n \psi(v_n))$.

V-rule If $\Gamma \vdash_V \varphi(\overline{m}/v_0)$ for every $m \in V$ then $\Gamma \vdash_V \forall v_0(\overline{M}(v_0) \rightarrow \varphi(v_0))$.

Note that a sentence is provable by the V-rule, in symbols $\vdash_V \varphi$, if $\Gamma \vdash_V \varphi$ for $T = \emptyset$.

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Semantics: Incompleteness

As far as Constraint 3 is concerned, we have the following:

Theorem (Incompleteness of Infinitary Logic)

Given any infinitary language $\mathcal{L}_{\kappa,\lambda}$, with $\lambda < \kappa$, and $\kappa \geq \omega_1$, for all sentences $\sigma, \Delta \in \mathcal{L}_{\kappa,\lambda}$, such that $\Delta \vdash \sigma$, if Δ is of *arbitrary* length, then $\models \sigma$ does not imply $\vdash \sigma$

The incompleteness of V-logic is a special case. We have that:

The 'Incompleteness Problem'

If V is uncountable, then there are Γ, φ such that $\Gamma \models_V \varphi \not\Rightarrow \Gamma \vdash_V \varphi$.

So, the logical incompleteness of V-logic leaves us with more models than proofs, and a disjoint syntax and semantics.

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Semantics: Incompleteness/Cont'd

Fact

If V is uncountable in our V-logic multiverse theory T, there is no 'real' outer model W s.t. $V \subseteq W$, that is, no V-logic *semantic* counterpart of the V-logic *theory* which asserts its existence.

So, if V is uncountable, Constraint 3 isn't met, and Constraint 2 is fully met *only syntactically*: we may *only* represent extensions of V through theories which assert their existence.

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Fixes

Fix 1 (Hyperuniverse): The easiest solution would be to assume the countability of V (V-logic is complete for V countable). However, this is *philosophically* problematic.

Fix 2: We content ourselves with (axiomatic) theories. This fix seems to fare better for various reasons, as:

- the multiverse will be developed without any appeal to 'intuition'
- we still have a neat articulation of multiverse membership
- historically, focus on axioms rather than on semantics has proved to be adequate in many ways

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One Further Fix: A Completeness Axiom?

Completeness

1) For every statement φ and for every outer model M of the ground universe, if $M \models \varphi$ then there is a proof of φ in V-logic.[3]

2) Any consistent V-logic theory T has models in V.

- This axiom will solve the 'incompleteness problem', ensuring the existence of a *proof* in V-logic of every purely *semantic* statement
- However, it is presently not clear how the axiom should be formulated so as to appear 'natural', and why it should be accepted

[3] More formally, $\forall\varphi, \forall M[\Gamma^M \models \varphi \implies \Gamma \vdash^M_V \varphi]$.

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Structure of the Presentation

Language and Axioms for $T_{\mathbb{V}_{mult}}$

A V-logic multiverse theory could thus be viewed as the collection of the following axioms:

1. Base Set Theory (BST)
2. (Width Multiverse) For all ψ, and $\varphi =$ "$\bar{W} \models \psi$" (where $\bar{V} \subseteq \bar{W}$), $Con(BST + \varphi)$
3. Further Axioms? E.g.: IMH (and refinements), Completeness, etc.

NB. The language is, as said, $\mathcal{L}_{\kappa^+,\omega}$, with individual constants: $\bar{V}$ for V and $\bar{W}$ for W, and infinitely many individual constants $\bar{a}$ for each $a \in V$.

Structure of the Presentation

Variants and Additions

- Add a height multiverse (consisting of top-end extensions of V)
- Use a stronger *infinitary logic*: $\mathcal{L}_{\kappa,\omega}$ with κ (at least) a *strongly inaccessible cardinal* (see next slide)
- Additional axioms: for instance, multiverse axioms such as IMH (maximality)
- Consider 'alternative' V-logics: for instance, if $V = L$, consider the L-logic multiverse: this looks like the *broadest possible* V-logic based multiverse concept one can have (as all universes compatible with L are also compatible with any extension of L)

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A Complete Infinitary Logic

- Consider V_ω-logic. This is equivalent to V-logic, only here V is just the rank initial segment V_ω
- This logic is complete (because of the ω-completeness theorem in $\mathcal{L}_{\omega_1,\omega}$)
- Now, consider the *next complete infinitary* logic $\mathcal{L}_{\kappa,\omega}$, where κ is, at least, strongly inaccessible.
- **Question**: is it possible to define a V_κ-logic based on $\mathcal{L}_{\kappa,\omega}$ which is also complete?

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Compatibility

The latter point leads to the following possible constraint/principle:

Constraint 4 (Compatible Universe Hypothesis [S. Friedman])

Given an extension of V, say, V^*, s.t. $V \subseteq V^*$, whenever there is a W extending V s.t. $W \models \varphi$, we have a corresponding W^*, extending V^* s.t. $W^* \models \varphi$.

The CUH asserts that, if we replace V with a larger V^*, the multiverse built around a bigger V^* does not decrease the set of truths compatible with V, that is, V^* has as many *compatible universes* as V.

CUH may also be viewed as an independent and new *maximality principle* for V (possibly leading to a characterisation of V as the 'maximal core' of the V-logic multiverse?).

Further Questions

- (Question 1) Consider a different base theory, such as: $T_1 = ZFC + LCs$, or $T_2 = ZF + AD$, etc. How would the V-logic multiverses built around T_1 and T_2 differ from each other? (Clue: use the notion of *compatibility* previously mentioned in connection with $V = L$)
- (Question 2) Consider a different V, with $V \neq L$. For instance, suppose $V = V_\kappa$, with κ a 'large' large cardinal. What would the V_κ-logic multiverse look like? (the question has connections with the mentioned goal of extending $\mathcal{L}_{\kappa,\omega}$)

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Thanks for your attention!

Antos, C., Barton, N., and Friedman, S.-D. (nd).
Universism and Extensions of V.
Pre-print.

Antos, C., Friedman, S.-D., Honzik, R., and Ternullo, C. (2015).
Multiverse Conceptions in Set Theory.
Synthese, 192(8):2463–2488.

Barton, N. and Friedman, S.-D. (2017).
Maximality and Ontology: how axiom content varies across philosophical frameworks.
Synthese, pages 1–27.

Barwise, J. (1975).
Admissible Sets and Structures.
Springer Verlag, Berlin.

Friedman, S. (2016).
Evidence for Set-Theoretic Truth and the Hyperuniverse Programme.
IfCoLog Journal of Logics and their Applications, 3(4):517–555.

Väänänen, J. (2014).
Multiverse Set Theory and Absolutely Undecidable Propositions.
In Kennedy, J., editor, *Interpreting Gödel. Critical Essays*, pages 180–205. Cambridge University Press, Cambridge.