# Week 6

## Notes on natural deduction for proposition logic

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Last class we introduced a natural deduction proof system for minimal logic. Although that system was already a good taste of what natural deduction is, it is not powerful enough for our need. Remember that the minimal logic system is composed by the following rules:

- the rules for conjunction: $\langle I\wedge\rangle$ and $\langle E\wedge\rangle$;
- the rules for disjunction: $\langle I\vee\rangle$ and $\langle E\vee\rangle$;
- the rules for implication: $\langle I\rightarrow\rangle$ and $\langle E\rightarrow\rangle$;
- and finally the rules for negation: $\langle I\neg\rangle$ and $\langle E\neg\rangle$.

It may seem that we do not need any more rules, since we have rules for introducing and eliminating all the connectives. For example, minimal logic is powerful enough to prove the following principle:

**(ECQ)** $\vdash_{ML} (\varphi \wedge \neg\varphi) \rightarrow \psi$.

This is the principle "*ex contraditione quodlibet*", i.e. "from contradiction, anything". Also, in the formula above I am not using the simple $\vdash$ symbol, but I am expanding it to note the natural deduction system I am using: in this case, $ML$ stands for Minimal Logic. This is because in this class we will use more than one natural deduction system (namely, minimal logic, intuitionistic propositional logic $IPL$ and classic propositional logic $PL$). Using only $\vdash$ would be then ambiguous.

Going back to the our $ECQ$ principle, here is its proof:

$$\dfrac{\dfrac{\dfrac{\langle \mathrm{E}\wedge\rangle\,\dfrac{[\varphi\wedge\neg\varphi]^1}{\varphi} \quad \langle \mathrm{E}\wedge\rangle\,\dfrac{[\varphi\wedge\neg\varphi]^1}{\neg\varphi} \quad [\psi]^2}{\bot}\,\langle \mathrm{E}\neg\rangle}{\neg\psi}\,\langle \mathrm{I}\neg\rangle,2}{(\varphi\wedge\neg\varphi)\rightarrow\neg\psi}\,\langle \mathrm{I}\rightarrow\rangle,1$$

Note in this proof that I have also numbered the rule in which the discharge of the assumption happens. This is going to be needed from now on as the discharge of assumptions becomes more common and we need to recognise which assumption we discharged with which rule.

An intuitive justification of the above principle is considering it like an argument, and try to see if it is valid. In this case, the premise $(\varphi \wedge \neg\varphi)$ cannot be true in any way, so the conclusion $\neg\psi$ cannot at the same time be false, *a fortiori*.

However, minimal logic is, in any case, "minimal". Most of what we need in logic is not provable with the rules of this simple system. For example, the Disjunctive Syllogism:

$$(\varphi \vee \psi), \neg\varphi \vdash \psi$$

is *not provable* in minimal logic: $(\varphi \vee \psi), \neg\varphi \nvdash_{ML} \psi$

To prove this principle, we need to add more power to our set of rules. Thus, we need the power of *at least* intuitionistic propositional logic. This is because in this new logic we are adding the following rule:

$$\dfrac{\bot}{\varphi}\,\langle \mathrm{E}\bot\rangle$$

This rule, called **Contradiction Elimination**, is quite powerful: if from a set of assumptions I can prove the contradiction, then I can prove *anything*!

Nevertheless, contradiction must be avoided at all costs: a too liberal use of this rule has paradoxical consequences. For example, the following argument would be valid:

(1) Logic is useless.
(2) Logic is not useless.

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(3) I am the richest person on Earth.

While this argument is trivially valid, it is obviously quite problematic. The corresponding proof in natural deduction is equally paradoxical:

$$\dfrac{\dfrac{\varphi \quad \neg\varphi}{\bot}\,\langle \mathrm{E}\neg\rangle}{\psi}\,\langle \mathrm{E}\bot\rangle$$

This is a proof of $\psi$ using as assumptions $\varphi$ and $\neg\varphi$, and seems too "easy". Nevertheless, it is useful in some situation.

In particular, it is the only way we can prove the Disjunctive Syllogism:

$$\dfrac{\varphi \vee \psi \quad [\psi]^1 \quad \dfrac{\dfrac{[\varphi]^1 \quad \neg\varphi}{\bot}\,\langle \mathrm{E}\neg\rangle}{\psi}\,\langle \mathrm{E}\bot\rangle}{\psi}\,\langle \mathrm{E}\vee\rangle,1$$

This intuitionistic propositional logic is then composed of all the rules of minimal logic, plus the Contradiction Elimination rule, and it is quite more powerful (as we have seen) than minimal logic.

However, $IPL$ is still not completely powerful. For example, while it can prove one way of De Morgan's Law (remember that De Morgan's Laws are the equivalences between conjunction and disjunction):

$$\vdash_{IPL} (\varphi \vee \psi) \to \neg(\neg\varphi \wedge \neg\psi)$$

it cannot prove the other way:

$$\nvdash_{IPL} \neg(\neg\varphi \wedge \neg\psi) \to (\varphi \vee \psi).$$

It is easy to see way. On the one hand, the former is an application of the Disjunction Elimination rule (just like the Disjunctive Syllogism). On the other hand, for the latter we need to derive $\varphi \vee \psi$ from $\neg(\neg\varphi \vee \neg\psi)$, and no matter how many tries, it is not possible to do it. The reason is that, with the rule that we have available, we can deduce $\varphi \vee \psi$ only by proving $\varphi$ or $\psi$ first (with the rules $\langle \mathrm{I}\vee\rangle$). But there is no rule to deduce $\varphi$ or $\psi$ from $\neg\varphi$ or $\neg\psi$ (the other way around would be possible even in minimal logic, with the rule $\langle \mathrm{I}\neg\rangle$).

How can we proceed then? As for the Disjunctive Syllogism, the only solution to our problem is to add a new rule to our system. Since intuitionistic propositional logic is not powerful enough, we need the full power of *classical* propositional logic. To achieve this power, we can choose between to rules. The first rule we could add is the **Double Negation eliminatation Rule**:

$$\dfrac{\neg\neg\varphi}{\varphi}\,\langle \mathrm{E}\neg\neg\rangle$$

This rule is already good enough: if we add it to $IPL$ we get classical propositional logic, and enjoy a lot more power in our derivations.

For example, take the following sentence: $(\neg(\varphi \to \psi)) \to \varphi$. This is a tautology, but we cannot prove it in intuitionistic propositional logic. This is because there is no rule in $IPL$ (or in $ML$) that can derive an arbitrary formula $\varphi$! Our only strategy would be to try to derive $\neg\neg\varphi$, and with our new rule we can deduce $\varphi$:

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg(\varphi \to \psi)]^1 \qquad \dfrac{\dfrac{\dfrac{[\varphi]^3 \quad [\neg\varphi]^2}{\bot}\langle \mathrm{E}\neg\rangle}{\psi}\langle \mathrm{E}\bot\rangle}{\varphi \to \psi}\langle \mathrm{I}\to\rangle,3}{\bot}\langle \mathrm{E}\neg\rangle}{\neg\neg\varphi}\langle \mathrm{I}\neg\rangle,2}{\varphi}\langle \mathrm{E}\neg\neg\rangle}{(\neg(\varphi \to \psi)) \to \varphi}\langle \mathrm{I}\to\rangle,\ 1$$

The other rule we can add, instead of E¬¬ (there is no need to add both of them, they would be redundant), is the **Classical Reductio ad Absurdum**:

$$\dfrac{\dfrac{\dfrac{[\neg\varphi]^n}{\vdots}}{\bot}}{\varphi}\langle \mathrm{CR}\rangle$$

This rule is actually very similar to the $\langle \mathrm{I}\neg\rangle$ rule, with the only difference that instead of deducing $\neg\varphi$ from a contradiction stemming from $\varphi$, we are doing the opposite, i.e. deducing $\varphi$ from a contradiction stemming from $\neg\varphi$.

Our previous derivation would look a little bit different:

$$\dfrac{\dfrac{\dfrac{[\neg(\varphi \to \psi)]^1 \qquad \dfrac{\dfrac{\dfrac{[\varphi]^3 \quad [\neg\varphi]^2}{\bot}\langle \mathrm{E}\neg\rangle}{\psi}\langle \mathrm{E}\bot\rangle}{\varphi \to \psi}\langle \mathrm{I}\to\rangle,3}{\bot}\langle \mathrm{E}\neg\rangle}{\varphi}\langle \mathrm{CR}\rangle}{(\neg(\varphi \to \psi)) \to \varphi}\langle \mathrm{I}\to\rangle,\ 1$$

As you can see, with the $\langle \mathrm{CR}\rangle$ rule we manage to derive our formula with one less step than with $\langle \mathrm{E}\neg\neg\rangle$. The reason is that classical reductio is *the most powerful* rule we can add to a natural deduction system (without adding paradoxical rules). However, it should be avoided in most cases, other than the one in which it is strictly necessary. Indeed, when using classical reductio in a derivation, is just like trying to kill a fly with a cannon. It could be done, but usually there are more elegant ways to do it.