Introduction to Logic [296.617]

Final exam

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1. For each of the following arguments, say if it is valid or not and, in each case, please justify your answer. [20 %]

a)

(1)	If the Moon is made of cheese, then Alan is right.
(2)	The Moon is made of cheese.
(3)	Alan is right.

b)

(1)	Every Greek is a man.
(2)	Socrates is a man.
(3)	Socrates is Greek.

c)

(1)	$A \to (B \wedge C)$
(2)	A
(3)	C

2. For each of the following sentences, use the truth-table method to find out whether it is contingent, a tautology, or a logical falsehood. [15 %]
   a) $(\varphi \wedge \psi) \to ((\varphi \vee \chi) \wedge (\psi \vee \chi))$
   b) $(\varphi \wedge \neg\psi) \wedge (\varphi \to \psi)$
   c) $(\varphi \vee \neg\psi) \leftrightarrow (\varphi \to \psi)$

3. Translate the following sentences: [20 %]
   a) **From English to the language of propositional logic**: If Mary loves Matthew but Matthew doesn't love Mary, then either Mary marries someone else and is happy or she doesn't and is unhappy.
   b) **From the language of first-order logic to English (please make use of the translation key provided)**: $((P(x) \to \neg D(x, y)) \wedge M(x, y)) \to \neg P(x)$
      - $P(x)$:= x is a prime number;
      - $D(x, y)$:= x is divisible by y;
      - $M(x, y)$:= x is a multiple of y;
   c) **From English to the language of first-order logic**: Every man that owns a house is happy.
   d) **From the language of first-order logic to English (again, please make use of the translation key provided)**: $\forall x[[L(x)] \to \exists y[M(y) \wedge \neg C(x, y)]]$, where
      - $L(x)$:= x loves climbing;
      - $M(x)$:= x is a mountain;

- $C(x, y)$:= x climbs y.

4. Consider the following sentence: [15 %]

$$\exists x \forall y [\exists z [R(z, z) \to R(x, y)]]$$

   a) Is it satisfiable? (provide an example)
   b) Is it valid? (if not, provide a counterexample)

5. Prove the following theorem using Natural Deduction Rules for propositional logic (primitive rules only): [15 %]

$$\vdash_{ND} (\varphi \wedge \psi) \to (\psi \wedge \varphi).$$

6. Prove one of the following sentences using Natural Deduction Rules for first-order logic (primitive rules only): [15 %]
   a) $\vdash_{ND} \exists x[A(x) \wedge B(x)] \to \exists x[A(x)]$;

Solutions

1. Arguments:
   a) Valid. The first premise is a conditional, while the second premise states that the antecedent is true. Then it cannot be that both premises are true and the conclusion is false.
   b) Invalid. There could be the case that Socrates is a man but not Greek.
   c) Valid. Just like the first one, with the difference that the consequent of the conditional in the first premise is a conjunction. If the whole conjunction is true (as in the argument), then the single components of the conjunction are true.

2. Semantic for propositional logic:
   a) Tautology:

φ	ψ	χ	$\varphi \wedge \psi$	$\varphi \vee \chi$	$\psi \vee \chi$	$(\varphi \vee \chi) \wedge (\psi \vee \chi)$	$(\varphi \wedge \psi) \to ((\varphi \vee \chi) \wedge (\psi \vee \chi))$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	0	0	1
0	1	1	0	1	1	1	1
0	1	0	0	0	1	0	1
0	0	1	0	1	1	1	1
0	0	0	0	0	0	0	1

   b) Contradiction

φ	ψ	$\neg\psi$	$\varphi \wedge \neg\psi$	$\varphi \to \psi$	$\varphi \wedge \neg\psi) \wedge (\varphi \to \psi)$
1	1	0	0	1	0
1	0	1	1	0	0
0	1	0	0	1	0
0	0	1	0	1	0

   c) Contingent

φ	ψ	$\neg\psi$	$\varphi \vee \neg\psi$	$\varphi \to \psi$	$(\varphi \vee \neg\psi) \leftrightarrow (\varphi \to \psi)$
1	1	0	1	1	1
1	0	1	1	0	0
0	1	0	0	1	0
0	0	1	1	1	1

3. Translations:
   a) $(L(x,y) \wedge \neg L(y,x)) \rightarrow (M(x,z) \wedge H(x)) \vee (\neg M(x,z) \wedge \neg H(x))$;
      i. $L(x,y)$:= x loves y;
      ii. $M(x,y)$:= x marries y;
      iii. $H(x)$:= x is happy;
   b) If x is a prime number then x is not divisible. But x is a multiple of y. Then x is not a prime number.
   c) $\forall x[(M(x) \wedge H(x)) \rightarrow Hp(x)]$.
      i. $M(x)$:= x is a man;
      ii. $H(x)$:= x is a house;
      iii. $Hp(x)$:= x is happy;
   d) For every man that loves climbing there is a mountain that he cannot climb.
4. First order semantics:
   a) $\exists x \forall y[\exists z[R(z,z) \rightarrow R(x,y)]]$ is satisfiable by the following structure $\langle \mathfrak{M}, s \rangle$:
      - Domain: $\mathcal{D} = \{1,2,3\}$;
      - $R^{\mathfrak{M}}(n,m) := n \leq m$, for all $n, m \in \mathcal{D}$;
      - $s(x) = 1, s(y) = 2, s(z) = 3$.
   b) On the other hand, $\exists x \forall y[\exists z[R(z,z) \rightarrow R(x,y)]]$ is not valid. Consider as a counterexample the same structure as before but with a different variable assignment:
      - Domain: $\mathcal{D} = \{1,2,3\}$;
      - $R^{\mathfrak{M}}(n,m) := n \leq m$, for all $n, m \in \mathcal{D}$;
      - $s'(x) = 3, s'(y) = 2, s'(z) = 1$.
5. Natural Deduction for propositional logic:

$$\dfrac{\dfrac{\dfrac{[\varphi \wedge \psi]^1}{\psi}\langle\wedge\mathrm{E}\rangle \qquad \dfrac{[\varphi \wedge \psi]^1}{\varphi}\langle\wedge\mathrm{E}\rangle}{\psi \wedge \varphi}\langle\wedge\mathrm{I}\rangle}{(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)}\langle\rightarrow\mathrm{I}\rangle$$

6. Natural Deduction for first order logic:
   a)

$$\dfrac{\dfrac{[\exists x[A(x) \wedge B(x)]]^1 \qquad \dfrac{\dfrac{[A(y) \wedge B(y)]^2}{A(y)}\langle\wedge\mathrm{E}\rangle}{\exists x[A(x)]}\langle\exists\mathrm{I}\rangle}{\exists x[A(x)]}\langle\exists\mathrm{E}\rangle}{\exists x[A(x) \wedge B(x)] \rightarrow \exists x[A(x)]}\langle\rightarrow\mathrm{I}\rangle$$