Towards an axiomatization of the multiverse conception of set theory

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1 Introduction

The importance of set theory can hardly be overestimated: from its first development by Cantor and Zermelo, to the most recent results, set theory has always been seen as providing a foundation for mathematics. And this is for an important reason: we can actually model and develop all of mathematics in set theory. Set-theory is usually identified with its standard axiomatisation, ZFC, that adequately captures our intuitive conception of sets and membership. The seminal work of Zermelo first suggested a 'cumulative' conception of set theory, according to which sets are constructed in stages and form a v-shaped structure, the universe of set theory V. However, from the sixties onwards it became clear that some propositions in the language of set theory cannot be proved from the canonical axioms of ZFC — the Continuum Hypothesis (CH), a central set-theoretic claim, being a case point. The independence of CH from ZFC strongly suggested that the usual set theory was inadequate as it stood. Whence the quest for finding an adequate extension of ZFC started: new axioms were proposed, but none of them was stably accepted as part of set theory. The reason is that several such new axioms are mutually exclusive: choosing one implies that all the results proved under an incompatible axiom can no longer be proven. This led to competing "set theories", and to the necessity to develop a formal methodology to compare the axiom candidates. At present, none of these theories is accepted as the "new" set theory.

One of the most recent development is the so called pluralistic conception of set theory. The advocate of this position considers set theory (and hence mathematics) not as a unique, single universe, but as a *multiverse*.[1] That is, the thought goes, the competing theories are all description of various universes inside the multiverse, and set theory, now taken in a broader sense, is the study of such a multiverse. Opposed to this position is the so called *universism*, the thesis that there exists only one universe of set theory, and adding new axioms to ZFC will eventually pin down which one is the "true" set theoretic universe.

The aim of this project is to argue in favour of a moderate pluralistic conception of set theory. To this end, I assume a *naturalistic* position in the philosophy of mathematics. According to Penelope Maddy, the main advocate of this conception, when tackling questions about the foundation of mathematics we should always take into consideration mathematical practice. For this reason, she proposes two principles aimed at guiding us in selecting the appropriate foundational framework for mathematics. The new theory should be foundational (UNIFY) and should prove as many as isomorphism types as possible. Together with Maddy's principles, I further take the ontological commitments of the mathematical theories accepted by mathematicians at face value. Thus, for instance, on the assumption that ZFC is naturalistically justified, since ZFC proves that there are sets, it follows that there simply are sets. And so on. A final requirement for the new foundational framework is to be recursively axiomatizable. That is, it must be possible in principle to tell within a finite time whether a given formula in the language of set theory belongs to the set of the set theoretic axioms. The standard motivation for the recursive axiomatisability of the axioms is that the axioms simply are principles for the correct use of the set theoretic vocabulary and that *we*, finite beings, are the ones who use such a vocabulary. Indeed, it can be argued taht the recursive axiomatisability of ZFC was one of the main contributing factors of its success within the mathematical community.

I argue that these premises are in tension with universism, i.e. that they support a form of pluralism about sets. In a nutshell, my argument is as follows. I agree with Maddy that her naturalistic principles justify ZFC. However, since set-theorists actually make use of *various models* of ZFC, it follows from my assumption that each of these models exist. Moreover, since these models are actually used by working mathematicians to pursue different research goals, we cannot simply treat them as "dreams" and "mistakes", to be eventually discarded once the mathematical community discovers the "true" model of the set theoretic universe. Rather, I propose that the various universes that compose the multiverse are all legitimate interpretations of the concept of set and of the membership relation $\in$, and set theory is the study of this multiverse of models.

However, such a pluralistic conception can be declined in different ways, which give rise to different multiverse conceptions of set theory. At one extreme we have Hamkins' *broad* multiverse,

[1] For a general account, see Caroline Antos et al. (2015).

in which there is no criterion to arrange all the universes; at the other the *vertical* multiverse, which is part of the Hyperuniverse program, with a very strong hierarchy.[2] Thus, rejecting the standard cumulative conception of set-theory, or any conception of sets according to which there is a single universe of sets, by no means settles the question concerning the nature and justification of the multiverse. In this project, I argue that the generic multiverse with a core developed in Steel (2014), is to be considered the best framework for mathematical practice. I argue that this is so for two reasons. To begin with, as I show, all the other conceptions of the multiverse are more or less deeply unsatisfactory. By contrast, the generic multiverse with a core (from now on GM_H) enjoys all the characteristics that made ZFC a successful foundation of mathematics and more. Or so I will argue. Accordingly, my project is structured in two main parts. In the first, I argue against various conceptions of the multiverse. In the second part, I further develop and study the GM_H, and I offer a naturalistic justification for it. In a nutshell, the arguments against the various multiverse conceptions are based on the naturalistic and realistic assumptions I introduced. For example, Hamkins' broad multiverse is too liberal, and it doesn't save the centrality that ZFC has in current set theoretic practice.

Secondly, I argue that the GM_H satisfies all the features that it is reasonable to expect from a framework for mathematical practice. First of all, since it has a core – a collection of statements that is true in every universe of the multiverse – the GM_H satisfies UNIFY and is, for this reason, foundational. Moreover, it encompasses our intuitive notion of set and membership. Second, unlike virtually all of its direct competitors, the GM_H is actually recursively axiomatisable. Just like ZFC, it provides a *conception* of sets, according to which the intended model of ZFC, the so-called cumulative hierarchy V, has multiple incompatible extensions. And it also provides a *theory* of sets, in the logician's sense of the term. Moreover, the GM_H satisfies MAXIMIZE in a better way than ZFC alone, since it is possible to prove that is has more isomorphisms types available. For all the reasons above I argue that the GM_H is the only possible choice as a framework for mathematical practice.

This project is structured as follows. First of all, a review of Maddy's principle UNIFY and MAXIMIZE is in order (section 3). After that, I present the classic set theory ZFC (section 2): first the axioms are informally described (section 2.1), then the classic set theoretic universe V generated from them is presented (section 2.2). The section closes with a discussion of the main reason that ZFC is not enough and has to be extend, the independence results (section 2.3). In my next step, I move on to discussing the multiverse conception for set theory will be discussed: after arguing that is safe from categoricity arguments (4.1.2), the various types of multiverse that have been build are presented (section 4.2). But in the following section (section 4.3) I argue that, other than the generic multiverse with a core, they are not satisfactory. In the next section (section 5) I explain why the GM_H is to be regarded as the most promising choice to be the framework for mathematics: first we describe how the GM_H is build, and then argue that it satisfies both the principles UNIFY and MAXIMIZE (section 5.1). In the conclusion (section 7), I sketch the road ahead. In particular, in section 7.2, I give a preliminary table of contents of my dissertation.

2 From ZFC to the multiverse conception of set theory

In this section, I outline the success story that is ZFC and classic set theory. First of all, I introduce ZFC, its axioms (section 2.1), and its underlying conception of set, viz. the *iterative conception* (section 2.2). Section 2.3 then introduces the main problem afflicting ZFC and its underlying conceptions of set: the so-called independence propositions, i.e. propositions that are logically independent from ZFC, such as the Continuum Hypothesis. More precisely, section 6.2.1 sketches the method of forcing, used to prove the independence of the CH, while section 2.3.2 briefly reviews some of the contemporary debate about CH.

[2]For a detailed account of the broad multiverse, see Hamkins (2012); for the Hyperuniverse program, see Friedman and Ternullo (2018a). For some details on this classification, see Koellner (2009).

2.1 Zermelo-Fraenkel with Choice: the basic set theory

Set theory is a mathematical theory devoted to the study of "sets" from a formal point of view. It is the main research field for anyone interested in the foundations of mathematics, that is, for anyone interested in the logical and philosophical foundation of mathematics.

Historically, set theory was first developed by Cantor and Dedekind in the late 19th century, while trying to formalize the concept of "set of points" and "set of reals". The linguistic and formal conventions (e.g. the quantifiers) developed in the same period by Frege, and the notation and the syntax developed by Peano, were soon incorporated in the theory, thus making possible a first axiomatization (by Zermelo in 1908, further developed by Fraenkel in 1922). This first axiomatic system (ZF) was later enhanced by Bernays, Skolem, von Neumann and Gödel, with the addition, among other things, of the Axiom of Choice (ZFC). After this first period, characterized by an optimistic development of what was considered *the* foundation of mathematics, Gödel first (with the incompleteness theorems) and Cohen after him (with the proof that the Continuum Hypothesis, CH, is not provable from ZFC) took away much of the optimism that accompanied the initial development of ZFC: set theory, as codified by ZFC, was clearly not enough. More recently, the development of forcing (by Cohen), inner models, and large cardinals gave new strength to ZFC, and now, at the beginning of the 21st century, set theory can hope again to become the "foundation" for mathematics.

The main point of interest of set theory is the fact that, from a very simple collection of axioms, it is possible to formalize the whole mathematics, from abstract algebra to chaos theory. Here are the axioms of ZFC:[3]

Axiom 1 (Extensionality) *Two sets are equal if and only if they have the same elements.*

Axiom 2 (Empty set) *There is a set with exactly no elements.*

Axiom 3 (Pair) *If A and B are sets, then there is a set that contains A and B (and only A and B) as elements.*

Axiom 4 (Union) *Given any set A, there is a set B such that, for any element c, c is an element of B if and only if there is a set D such that c is a member of D and D is a member of A.*

Axiom 5 (Infinity) *There is an infinite set, i.e. there is a set with infinitely many elements.*

Axiom 6 (Power) *For every set A there exists a set B such that the elements of B are exactly the subsets of A.*

Axiom 7 (Foundation) *No set is an elements of itself.*

Axiom 8 (Separation) *Given any set A, there is a set B such that, given any set x, x is a member of B if and only if x is a member of A and the predicate φ holds for x.*

Axiom 9 (Replacement) *The image of any set under any definable function is also a set.*

Axiom 10 (Choice) *Given a non-empty collection of sets, is possible to chose an element from every set of the collection.*

From this collection of axioms, it is possible to model all the known mathematics, and to develop a universe of sets that goes from the empty set to infinite sets: the cumulative hierarchy.

[3]Strictly speaking, Axiom 8 is an *axiom schema*, generating infinitely many axioms, one for each predicate φ in the language of set theory.

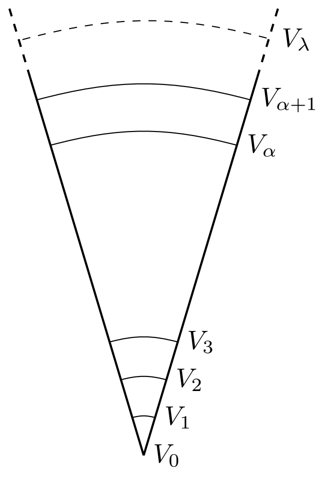

Figure 1: The cumulative hierarchy

2.2 The universe of ZFC: the cumulative hierarchy

The universe of set theory (the universe of all sets) is usually referred to as the *cumulative hierarchy*, or von Neumann's hierarchy.

The existence and the uniqueness of the general transfinite recursive definition of sets, that is the foundation of this hierarchy, was proved by von Neumann in 1928. With this proof, the hope for a formal foundation of mathematics reached its peak: not only the axioms were perfect in modeling all sorts of mathematical objects, but the universe of all sets (and so, the universe of mathematics) was shown to be unique. However, looking at that period with some prospective, we can actually recognize the first seed of the future development of the multiverse. In fact, von Neumann proved not only that V was unique for ZFC, but also that his personal axiomatization of set theory (a slightly different axiomatic system than ZFC) has a unique universe. And, the most important thing, he did not prove that these two universes were the same. Von Neumann's paper not only marked the birth of the cumulative hierarchy, but also that of the set theoretic multiverse.

So, how is this universe built? First, we consider only one set: the empy set. This is the first level of the hierarchy. Then, every new finite level is constructed using the union and powerset operation. When we reach the limit level we take the union of all the previous not limit levels. In this way, we can prove for any set that it is in a level of the hierarchy. We can define recursively this process (also, cfr. Figure 1):

$$V_0 = \emptyset;$$
$$V_{\alpha+1} = \mathcal{P}(V_\alpha);$$
$$V_\lambda = \bigcup V_\alpha \text{ (for all } \alpha < \lambda\text{, where } \lambda \text{ is a limit ordinal)}$$

V fully embodies the intuitive and commonly accepted *iterative* concept of set (see Boolos (1971)).

The story so far has much to recommend: we have an axiomatic system (ZFC) strong enough to represent virtually every currently accepted mathematical theory and object, with a strong motivation behind it, the cumulative hierarchy. Sadly, however, there is a fly in the ointment: Gödel proved that any system that interprets a modicum of arithmetic (and ZFC is certainly one such) is incomplete, in the sense that it does not decide sentences formulated in its language, such as, for instance, a sentence to the effect that such a system is consistent. Since such sentences can be seen to be true by means of set-theoretic reasoning carried out outside of ZFC, ZFC, and indeed any system satisfying Gödel's minimal conditions, can be seen to be in some sense inadequate. However, it was still believed that set theory was still powerful enough to prove any other mathematical result (other than its own consistency). The blow against this belief was struck during the sixties: it was shown that the most important open question in set theory, the Continuum Hypothesis, cannot be solved within ZFC.

2.3 The Continuum Hypotheses and its independence

The Continuum Hypothesis (CH) is probably the main reason behind the development of extensions of ZFC and of the multiverse itself. Section 2.3.1 introduces the CH. Section 6.2.1 will lather briefly introduce the method of forcing, used to prove its independence.

2.3.1 The Continuum Hypotheses

The CH is a claim about the size of sets: it states that there is no set with more elements than the set of natural numbers $\mathbb{N}$ *and* fewer elements than the set of real numbers $\mathbb{R}$. Despite its simplicity, this statement has important consequences: if CH is false, then we could build a set S with no bijection with neither the natural numbers nor with the real numbers. This means that some element (actually, an infinite amount of them) of S will always be left out and, at the same time, we would have the usual behavior between S and $\mathbb{R}$ (with an infinite amount of real numbers that are not elements of S), thus getting a situation like this: $\mathbb{N} < S < \mathbb{R}$. And yet, given the iterative process of the cumulative hierarchy we know that, using only the operations used to build up V, there is no way a set such as S could be build in the first place. More generally, CH is fundamental for our understanding of infinite sets and of their arithmetical properties (this is especially true of the so-called Generalized Continuum Hypothesis, GCH, concerning sets of arbitrary size). In particular, without CH is very difficult to define the exponentiation between cardinals.

The Continuum Hypothesis is probably the most important and famous open problem in set theory. It was first conjectured by Cantor, in the late 19th century, and was the first problem of Hilbert's Millennium Problems. The importance of the CH was immediately understood in the mathematical community, and a lot of effort was put in trying to prove it from ZFC. However, it soon became clear that the challenge was impossible: in 1963 Paul Cohen showed how to build a model of ZFC where the CH was true *and* a model of ZFC where the CH was indeed *false*, thus proving that the CH was *independent* from the axiom of ZFC. At this point the "multiverse", until now formed by only two universes (the canonical V and the universe from the von Neumann's axiomatization) was doubled: we had two universes based on the canonical V (the one with the CH true and the one with the CH false) and two universes based on von Neumann's axiomatization (as before). From that moment, set theory underwent a deep transformation: the main effort was in building more of these models (universes), each different from the others, and in enhancing the canonical set of axioms ZFC so that CH would become provable in it.

So what could be a solution to the continuum problem? What would be needed is either a proof of CH or of its negation. Since no such proof can be obtained in ZFC, the focus is now shifted to adding new axioms that imply the CH, thus proving it, or that imply that the CH is false. There are several candidate extra axioms, many of which are incompatible. The upshot, then, is a number of extensions of ZFC, many of which are mutually incompatible. This presents us with a choice: electing one of these theories as "the one" set theory, thus replacing V and scraping all the other theories (and universes), or building a multiverse, where all these theories are on the same "plane".

2.3.2 Status of research

A complete overview of the enormous bibliography on the Continuum Hypothesis goes beyond the scope of the present proposal. In what follows, I simply point to the most recent and important results.

First of all two papers by Solomon Feferman are worth mentioning: (Feferman (2009) and Feferman (2011)), both of which argue *against* the definiteness and importance of the CH. Albeit controversial, both papers defend a naturalistic approach to CH (which is usually approached to on philosophical grounds, and seldom on mathematical ones). This means defining and justifying the CH with particular attention to the needs of mathematical practice instead of focusing on purely philosophical arguments.

Regarding the possible solutions to the problem of the continuum, Freiling (1986) contains a philosophical survey (with just enough mathematics). Woodin (Woodin (2001a), Woodin (2001b), Woodin (2005) offer more substantial contributions, together with the already cited Woodin (2011)) and Shelah (Shelah (2000)). In particular, while Woodin's work is mostly against the CH (i.e. he thinks that the hypothesis is actually false), Shelah proves the first *positive* results about the CH (and indeed the GCH) in almost fifty years.

3 The naturalistic approach in the philosophy of mathematics

In order to critically assess the various options for dealing with claims such as the CH that are independent of ZFC, I assume that the best philosophical guide available is mathematical practice itself. That is, I assume a form of *naturalism* about mathematics.

Naturalism in the philosophy of mathematics is a methodological approach that recommends using mathematical rules and arguments to justify mathematical objects (usually axioms) and statements about mathematics. In most cases, this means that a new axiom (or theory) should be tested on the basis of its actual or potential benefits for mathematical practice. Every single argument about mathematics should have a specific mathematical goal. To put this definition in a more precise fashion, Penelope Maddy proposes the following principles as a method to test the strength of new axioms (and, subsequently, theories)[4]:

Principle 1 (UNIFY) *The ultimate goal should be a theory where every structure and every mathematical object can be modelled.*

This essentially says that an adequate set theory should be foundational, in the sense that it should allow one to model most (better: all) of current mathematical practice.

Principle 2 (MAXIMIZE) *The new axioms should be as powerful as possible, it should* maximize *the range of available isomorphism types.*

The intuitive idea behind this second principle is that a foundational framawork should allow mathematicians to compare different objects as described in different theories and their relationships. And the more isomorphisms a theory can prove, i.e. the more relations of 'structural identity' between structures and objects can be established, the more a theory can fulfill this foundational goal.

Following Maddy (2017), we can refine our definition of UNIFY, saying that a theory T is foundational if and only if it provides the following:

Meta-mathematical Corral A theory T should make it possible to embed all of mathematics in a single theory, that is, we should be able to prove, in our foundational theory, something general about all mathematics.

Elucidation The foundational theory T should be able to replace a vague notion with a more precise one. For example, consider the notion of continuity which, before Dedekind's work, was somewhat ambiguous. It was precise enough to generate the calculus, but not enough to become a tool in the proof of fundamental theorems. Only after it was defined in set-theoretic terms it became certain and useful for any purpose.

Shared Standard Our foundational theory T should provide a Shared Standard to decide if a proof is actually part of mathematics or not. In other words, our foundational theory T should be the "judge" of what counts as a proof in mathematics: a formal derivation in the foundational theory should be regarded as the "standard" proof in mathematics.

Risk Assessment We should be able to use our foundational theory to assess any particular new mathematical object considered dangerous and suspicious, and to determine how risky it is to use it.

Furthermore, we can rephrase the definition of MAXIMIZE, still following Maddy (2017)[5], in a way that relates it more to the Shared Standard, as follows:

[4] See Maddy (1997) and Maddy (2011).

[5] However, slightly changing it: while Maddy consider Generous Arena one of the foundationility features of UNIFY, I prefer to put it in correlation with MAXIMIZE, since it shares the same goal and phrasing.

Generous Arena The foundational theory T should be a Generous Arena in which we can hope to find, study and analyze every mathematical object.

This means that all the various structures studied in all mathematical branches can co-exist in it, and their interrelations studied. Moreover, our foundational theory should be a place where is possible for a mathematical field to borrow methods from another field. Note that the isomorphisms part is still contained in this definition: if all the various structures from mathematics can co-exist in a single framework, the total number of isomorphisms grows.

Maddy, still in her Maddy (2017), argues that these principles justify the classic set theory ZFC with the Single Universe V as the currently best framework that could be considered a foundation for mathematical practice:

> In sum, then, it seems to me that the familiar set-theoretic foundations, rough and ready as they are, remain the best tool we have for the various important foundational jobs we want done. (Maddy (2017, p.53))

In a nutshell, her argument is that ZFC satisfies UNIFY (it is well known that all of the known mathematics can be represented within ZFC) and that it also scores well on MAXIMIZE. Moreover, the rival theory that Maddy considers, category theory, doesn't score as well as set theory with respect to UNIFY. Her main argument here is that category theory (with some additions) can at best hope to be just as good as ZFC, but without any real advantage. For one thing, any advantage category theory might be thought to have over ZFC vanishes as soon as one considers the theory ZFC + etc. Secondly, Maddy points out that category theory lacks the intuitive justification of its axioms enjoyed by ZFC. In particular, whereas the axioms of ZFC are justified by the iterative conception of sets, no analogous conception of categories appears to be available in the case of category theory (for details, see Maddy (2017)).

Pace Maddy, I will argue that a certain theory of the multiverse, and its corresponding conception of sets, the generic multiverse with a core, actually score better than ZFC with respect to Maddy's criteria. In section 5.1 I argue that a particular kind of multiverse, the generic multiverse with a core, fares equally good than ZFC regarding UNIFY, and is actually better regarding MAXIMIZE.

4 The Multiverse

In this section, I outline the multiverse conception of set theory. Section 4.1.2 argues that the multiverse conception is is not undermined by categoricity arguments.. Section 4.2 introduces various conceptions of the multiverse. Section 4.3 argues that each of these conceptions is in some respect problematic.

4.1 Against universism and categoricity

I first offer a general argument (from Maddy's principle MAXIMIZE) in favour of the multiverse. I then consider a potentially strong objection against multiverse conceptions of set theory, i.e. that once properly formalised set theory has only one model up to isomorphism. I suggest, however, that the required *second-order* formalisation is improper.

4.1.1 Against universism

As stated in the introduction, universism is the thesis that there is only one set theoretic universe, V. This universe is the so called "canonical" model of set theory, as opposed to all the others models, the *non-standard* models. For example, the constructible universe L is a non-standard model of set theory.

An argument against pluralism is that in the single universe V we can actually simulate every non standard models of set theory. For instance, in ZFC we can simulate a model of $ZFC+V=L$ or a model of $ZFC+LCs$, even though they are incompatible.[6] However, we cannot simulate them

[6] $V = L$ is the Axiom of Constructability, that says that all the sets of the universe can be build from simpler sets, and it is incompatible with the existence of most large cardinals (LCs).

at the same time. This means that in V we can have a simulation of $ZFC+V=L$ in the canonical model, but then, when trying to simulate $ZFC+LCs$ we are forced to throw away all we proved in the simulation of $ZFC+V=L$. The main consequence of this fact is that we cannot compare two non standard models at the same time. By contrast, all the different models are available in the multiverse, at the same time, and we can prove isomorphisms between their structures. Thus, compared with a pluralistic conception of set theory, we are loosing the ability to investigate those models synchronically. This means that, when comparing a single universe prospective with a multiverse prospective using the principle MAXIMIZE, the latter will fare better than the former. Consequently, from a naturalistic point of view, a multiverse conception of set theory is a more Generous Arena than the Single Universe.

4.1.2 Against categoricity

Possibly the strongest argument against the pluralistic conception of set theory is that, once correctly formalised, set theory *has only one model*. More specifically, the argument assumes that set theory must be formulated at the *second order*. That is, set theory must not only quantify on *members* of the domain, but also on *arbitrary subsets* of the domain. The main consequence of a second order formulation is that that theory becomes categorical: it becomes possible to prove that it has only one model – or, more technically, that all the models of that theory are isomorphic. This is true, for example, for second order arithmetic (PA_2, which is just the same as PA, plus the comprehension axiom and the second order induction schema). We can prove a similar (albeit somewhat weaker) result also for the second order formulation of set theory, ZFC_2: Given any two full models of ZFC_2, either they are isomorphic, or one is isomorphic to a proper initial segment of the other.[7] This result entails that, in ZFC_2, many independent statements such as the CH have a *determinate* truth-value. Since the multiverse is founded on the existence of more than one model of set theory, the pluralist cannot accept second-order axiomatisations of set theory. But are such axiomatisations acceptable in the first place?

McGee and Martin have tried to revamp this argument, arguing that determinacy it is not a feature of the logic used to formulate the theory, but a consequence of the axioms itself. To this end, McGee (1997) proves that second order set theory with urelements[8] ($ZFCU_2$) is also categorical, while Martin (2001) argues that the notions of set and membership are unique and thus can be represented only by a categorical axiomatization.

The problem with these two arguments is that they are both based on very strong assumptions: McGee's theorem is based upon the Urelements Axiom (that says that there exist urelements), while Martin arguments are based on the Uniqueness Postulate (a form of extensionality that applies not only in a structure, but *across all structures*). It can be argued that both assumptions have an ad hoc character: while they help proving categoricity results, they are not really used by mathematicians. More precisely, they are not needed to prove standard mathematical results.[9]

More generally, the main problem with categoricity results is that the concept of set is an *algebraic concept*. Briefly, this means that set theory is no different from group theory or any other algebraic theory. The main intention when formulating these theories is to admit non isomorphic models, since mathematicians need different models for different purposes[10]. All these models are considered totally legitimate by the majority of working mathematicians. The main consequence of this attitude is that second-order determinacy is seen by most set-theorists simply as a consequence of the in-built *logical* features of the theory, rather than as a consequence of the second-order axioms' ability to express a more determinate concept of set. Thus, set-theoretic indeterminacy has progressively come to be viewed as an integral part and stable feature of the current set-theoretic landscape.[11]

[7] This was proved by Zermelo (1930), for an in depth discussion of categoricity see Button and Walsh (2016).

[8] An urelement is an object that it is not a set, but could be a member of a set.

[9] In particular, Martin's Uniqueness Postulate is exceedingly strong: it allows one to prove that even *first-order* set theory is categorical!

[10] For example, consider the hierarchy of large cardinals, as superbly described in Kanamori (2008).

[11] As argued by Mostowski (1967) and, more recently, Hamkins (2012).

4.2 Various multiverse conceptions

As we have just seen, the concept of "multiverse" was born following the discovery of the phenomenon of independence in set theory: sentences in the language of set theory, such as the Continuum Hypothesis (CH), that turned to be independent from the axioms of ZFC. In order to prove these independence results, set theorists make use of *models* (universes) different from the canonical one: the collection of all these models (universes) constitutes the multiverse.

The multiverse then consists of all the models that satisfy the axioms of any set theory (say, $ZFC, ZF + \neg C, ZF + AD, \dots$). In addition, these models contain all the relevant information (although sometimes mutually incompatible) on sets. Each of these models is a legitimate universe of set theory, so there is no Single Universe. According to proponents of the multiverse, this lack of unity cannot be repaired in any way and set theory is precisely the study of these alternative universes, in which the properties of sets can vary greatly from one to another.

From a philosophical point of view, we can classify the various types of multiverse by their commitment to the ontological existence of the universes. More precisely, we can recognize two main positions:

- a *realist* multiverse, committed to the full existence of the universes that form it (e.g. Balaguer's fullblooded Platonism[12]);
- an *anti-realist* multiverse, that does not commit to the platonic existence of the universes (this is, for instance, the position defended by Shelah in Shelah (2002) and Shelah (2003)).

Instead of focusing on the existence of the universes, the mathematical classification is based on how we build a hierarchy of all these universes. So we can have:

- the *broad* multiverse, where there is no hierarchy at all, and all the universes have the same status among the others (for instance, Hamkins' multiverse is a broad multiverse[13]);
- a *generic* multiverse, where we differentiate between universes using a strong logic (an idea owed to Woodin[14], from now on GM_Ω) or supposing the existence of a *core* (an idea owed to Steel[15], that is the GM_H);
- a *vertical* (or *horizontal*) multiverse, where all the universes are like on a ladder, bigger and bigger (or on the same level, but we add more and more universes of the same size);
- a *parallel* multiverse, where all the universes are hidden from each other (Väänänen's multiverse is of this kind, cfr. Väänänen (2014));
- the *hyperuniverse*, that is, the collection of all the countable transitive models of ZFC (Friedman and Arrigoni's construction[16]).

One can of course apply the philosophical classification to the mathematical one, which yields a realist and anti-realist version of all these multiverses.

Not all these multiverses are equally appealing. Some are philosophically weak; others are mathematically trivial, and, as we will see, most of them do not satisfy the desiderata introduced in section 1. I will argue in Section 5 that, by contrast, these desiderata are all satisfied by the generic multiverse with a core.

4.2.1 Status of research

The debate about the set theoretic multiverse is presently roaring. In addition to the work of Steel (that will be presented in section 5), the last few years have seen the emergence of interest in the philosophical assumptions behind the multiverse. On this topic there are two notable contributions: a paper from Koellner (Koellner (2009)) and a paper from Antos and Friedman (Caroline Antos et

[12]Cfr. Balaguer (1995) and Balaguer (1998).
[13]Cfr. Hamkins (2012).
[14]In Woodin (2011).
[15]Cfr. Steel (2014).
[16]Cfr. Arrigoni and Friedman (2013).

al. (2015)). The latter paper is also a follow up to Arrigoni and Friedman (2013) on the hyperverse. While the latter is more technical and focused on the mathematical construction of the hyperverse, Antos and Friedman's contribution is more philosophical, and also involves the construction of a new version of the multiverse, the so-called vertical multiverse. Antos and Friedman's paper also contains a helpful classification of the various multiverse positions. Finally, a number of articles recommending and articulating specific conceptions of the multiverse have been recently published. To mention but a few, Koellner (2013), Gitman and Hamkins (2010) and Hamkins (2012) focus on the Hamkins' broad multiverse, while Woodin (2011) articulates Woodin's GM_{Ω}. The most important voice against the multiverse is probably Martin (2001), who advocates a single universe on categoricity grounds. The main problem of Martin's defense of the Single Universe is that his argument heavily relies on very strong, and ultimately too restrictive, assumptions, (more specifically, these assumptions restrict which sets we are allowed to form, in ways that significantly go beyond the restriction already present in ZFC).

The philosophical debate on the foregoing mathematical results is also booming: Balaguer provides the philosophical foundation for Hamkins' multiverse (Balaguer (1995) and Balaguer (1998)), Koellner, Shapiro, Shelah and Wang provides more general reflection on pluralism and the development of set theory (Koellner (2009), Shapiro (2000), Shelah (2002), Shelah (2003) and Wang (1996)). Still unpublished, there is also a contribution by Maddy[17] focused on the foundationality of the multiverse.

4.3 Arguments against some multiverse conceptions

In this section I will argue that all the multiverse conceptions considered until now are not the viable choices to be the new framework for mathematics. As a reminder, I compare them considering the assumptions of naturalism and simple realism introduced in section 1 and section 3.

We have already seen that a form of pluralism is more than plausible in set theory. This is because there is no agreement between mathematicians on how to continue the cumulative the set theoretic hierarchy into the higher infinite (and even if there is a higher infinite: it is possible to defend the Axiom of Constructability that is incompatible with most of the large cardinals). Nevertheless, the higher infinite is mathematically investigated, and most of the recent results in set theory are conditional on the existence of some large cardinal. The focus is now shifted on which pluralistic conception complies with our minimal assumption of naturalism and simple realism. A radical form of pluralism is going to be problematic: as we will see in section **??**, if our pluralism is to radical our conception is not going to be axiomatizable or foundational. Moreover, we would have problems recognizing our intuitive notion of set and membership. In the next sections (section 4.3.2, section 4.3.3 and section 4.3.4), we will see that even more moderate conceptions of pluralism have the same kind of problems, in addition to problems in respecting our simple realism.

4.3.1 The Broad Multiverse

Hamkins' *broad* multiverse (for a detailed account see Hamkins (2012) and Koellner (2013)) is the most radical and liberal: the universes are not arranged in any hierarchical order, they all have the same status among the others. Moreover, there is no criterion to select between universes, so every imaginable universe is part of this kind of multiverse. In Hamkins' view, the main job of set theory is to deal with different kinds of constructions, which can verify, or falsify, the same set-theoretic claims (such as CH). In such a framework, there is no reason to ban a particular universe: they are all perfectly legitimate model theoretic constructions, and thus part of the multiverse. The various universes are the all the models of a given (consistent) theory T. From these universes (models), we can build even more universes (models) using set forcing. To see this, suppose T generates a universe W. We can then apply set forcing on this universe W to generate various generic extensions $V_i[W]$, and we can then iterate this process. All these universes, both the ones generated from the theories, and the ones generated with set forcing, are part of the broad multiverse.

Hamkins' universe is very *rich*: virtually every model of every consistent theory in the language of set theory is part of such a universe. As such, Hamkins' universe can be thought to be a

[17] To appear in A. Caicedo et al., eds., *Foundations of Mathematics*, Providence, RI: AMS.

maximally general framework for mathematics—one that doesn't impose any constraints on our conception of sets. However, this is precisely what I plan to argue is the main problem with Hamkins' universe. Hamkins' conception falls short on foundationality. More precisely, Hamkins' multiverse doesn't satisfy the foundationality features of UNIFY (described in section 3), namely Meta-mathematical corral, Risk assessment, Elucidation and Shared Standard. The main reason is that there is no "bridge" between universes: every universe is isolated from the others, with its particular concept of set and membership. In Hamkins' view, every universe satisfies a different set of axioms and principles and the concept of set and membership is tied to these axioms. For example, we could have a universe without extensionality, in which the concept of membership is deeply different to the classical one. Obviously it would be very difficult (if not impossible) to define a common notion of set between the two universes, since in one universe sets are determined solely by their members (the classic case), while in the other sets are determined in a different manner.

The problem of Hamkins' conception is, then, that it doesn't provide a common language for the metatheory. First of all, we should say that the universes of the broad multiverse are all the possible interpretations of the language of set theory. For example, we have a universe in which the membership relation is interpreted in the classical way and another in which is not extensional. Obviously these two universes are mutually incompatible, and we cannot define a common language for our metatheory when talking about them, since they both interpret the language in very different ways. For this reason, Hamkins cannot define a metatheory: to avoid limiting the possible universes, his metatheory is simply first order logic with a binary relation, $\in$, whose interpretation is completely left open.. His argument seems very naturalistic: the intended interpretation, or range of interpretations, of the language of set theory is. He leaves this task to mathematicians. He limits himself to providing the most general and abstract language and leaves the task of interpreting it to the mathematicians. The problem lies in the fact that this uninterpreted metatheory and the presence of every possible universe in the multiverse makes impossible any metatheoretic discourse. In fact, since our binary relation is not interpreted in our metatheory, when studying a particular universe, we have to resort to use the interpretation of that particular theory as the interpretation of our metatheory. For example, when studying a universe in which the membership relation is classical, our metatheoretic binary relation will be the classical membership relation. But then, when trying to study a different universe, we find out that our metatheory is incompatible with our new object language, since our metatheory language is taken from the previous universe. For example, if we try to study a universe in which the sets are not well-founded, then we find that our classical membership relation that we are using in the metatheory is not the same as the membership relation in the object theory. The only way to avoid this is to define a second metatheory, to talk about the other universe, in which the two relations are interpreted in the same manner. We have no choice but to iterate the process indefinitely: for every universe, a different metatheory must be defined.

The main consequence of all of this is that the broad multiverse cannot satisfy any of the foundationality features classified by Maddy. First of all, to be able to embed all mathematics in our multiverse we would need at least a common language, a common concept of set and a common concept of set. Since we cannot assume that the language is the same across all the multiverse, we cannot have the same concept of set and membership in all the multiverse. Thus, the way mathematics is embedded can greatly vary from one universe to another. For the same reasons, we cannot rely on the multiverse to elucidate mathematical objects, since the definition of an object in one universe could be totally different in another universe (it can even be the case that the object in another universe doesn't exist at all!). This problem is even more pressing when considering the ability to assess the risks arising from the use of a certain mathematical object: our assessment would be quite weak if we base our reasoning on a multiverse where the object we are studying can exist in a way in one universe, and in the opposite way in another. Finally, the aforementioned difficulties in deciding the background theory will make it difficult to decide a shared notion of proof across all the multiverse. Consequently, we cannot consider Hamkins' broad multiverse a good candidate to be the foundational framework for mathematical practice.

4.3.2 Woodin's generic multiverse GM_Ω

Woodin's multiverse GM_Ω[18] is a type of generic multiverse: it is generated applying set generic forcing to enlarge a ground model. This kind of multiverse is closed both under (non trivial) extensions of the ground model (enlargements) and by refinements of it (inner models). Moreover, Woodin defines two Multiverse Laws: without getting in too technical details, they state that the multiverse truth cannot be recursively defined in a particular set (universe) of the multiverse.

Woodin uses this particular construction to argue *against* a multiverse conception of set theory. In particular, he claims that the set generic multiverse he defined violates both Multiverse Laws. In other words, Woodin claims that the main feature of a multiverse conception should be that its truth cannot be defined in a particular universe of the multiverse itself. However, if the Ω Conjecture is true (a statement proposed by Woodin in his infinitary Ω Logic), and if there is a proper class of Woodin cardinals (a particular kind of large cardinals), then the truth of the generic multiverse GM_Ω is recursively definable in a set of it.

The problem is that it is not very naturalistic. In particular, Woodin doesn't justify his Multiverse Laws, and the claim (central for his argument) that the multiverse conception of truth should not be recursively definable in a universe of the multiverse. Moreover, his argument is based upon very strong mathematical assumptions – namely the Ω Conjecture – that are too specific and difficult to endorse. In particular, since the Ω Conjecture implies that the CH is false, it restricts arbitrarily the scope of the multiverse. Moreover, since the assumptions needed to let the multiverse work are so specific, it means that the GM_Ω doesn't comply with our intuition of the notion of set and membership.

4.3.3 The Parallel Multiverse

Väänänem defines his multiverse[19] as the cumulative collection of all the possible parallel universes. To build this universes, he chooses to use different notions of the power set operation, thus in his multiverse it is possible to have two power set from the same set, $\mathcal{P}_1(X)$ and $\mathcal{P}_2(X)$. Then he defines some axioms and a logic to make sense of this multiverse, in particular to be able to differentiate between all the parallel universes.

Väänänem's parallel multiverse is apparently quite good, since it is axiomatizable, it seems quite naturalistic and doesn't have problem with our simple realism. But, in reality, has two main problems. First of all, it is still very underdeveloped, so we cannot really know all the consequences of its adoption. In particular, it doesn't address a clearly defined collection of structures which may count as universes, since it's construction doesn't have a variable for the universes. This also means that all the universes are hidden from each other, and we cannot access any other universe while being in one. This negates all the possibilities to actually satisfies naturalism, and thus its adoption remain quite problematic.

4.3.4 The Hyperuniverse Program

Friedman and his research group[20] define the Hyperuniverse as the collection of all countable transitive models of ZFC. Moreover, they define a new V-Logic to investigate it, with particular attention to the phenomenon of maximality.

Friedman's Hyperuniverse while being a promising research program, that managed to prove some interesting mathematical results, nevertheless it has some problems. First of all, it is not an axiomatic theory, but this can maybe be solvable. The main problem is that it defines the multiverse has the collection of all the countable transitive models of ZFC. This restrict a lot the scope of the multiverse, and doesn't comply with our simple realism, since it describes the existence of mathematical objects without committing to their existence. Furthermore, he proposed two hypotheses, the Inner Model Hypothesis (IMH) and the Countable Inner Model Hypothesis (CIMH). While both of them are very helpful in investigating the maximality of the Hyperuniverse, they both imply the failure of CH (and GCH) and the non existence of large cardinals. These facts

[18]See Woodin (2011).
[19]See Väänänen (2014).
[20]See Friedman (2018).

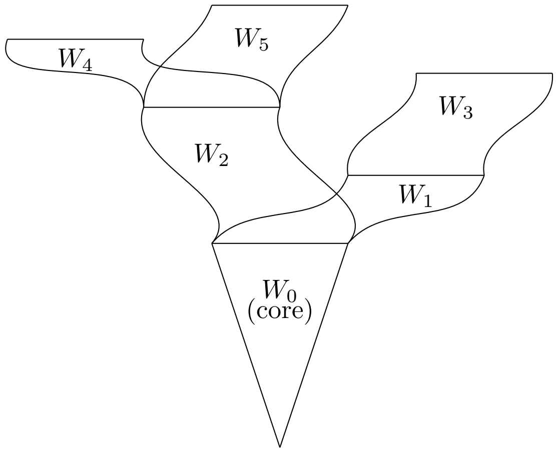

Figure 2: The generic multiverse with a core

seem particular problematic, since restrict the multiverse arbitrarily and go against mathematical practice (that makes very good use of large cardinals).

In conclusion, while the multiverse conception is, the most natural way to extend V to get a new framework for mathematics, no multiverse conception considered until now seems to be enough for the job. Since they all fail to satisfy our minimal assumption of naturalism and simple realism, they can all be regarded as less apt to the job than ZFC itself. But not all of them: in the next section I will outline one multiverse conception that seems to be what we need to replace V.

5 The Generic Multiverse with a core (GM_H)

The generic multiverse with a core (from now on GM_H), developed by Steel in Steel (2014), is one of the most promising candidates between all the multiverse conceptions, since it is the only one that satisfies all our assumptions stated in section 1. The main goal of this construction is to *maximize interpretative power*: we should develop a foundational language and theory that should allow us to prove as much results as possible in the most efficient way. In other words, in our foundational language and theory we should be able to develop all present, and possible future, mathematics.

The main feature of the GM_H is the fact that it has a *common core*. This common core is shared between all the universes of the multiverse: it is formed by all the statements that are true in every universe of the multiverse. Intuitively, such a multiverse is like a tree: the trunk is the core, while the universes are the leaves. Every leaf is unique, but it is deeply connected with all the others by a "common ground" (see figure 2). In this section I will present the axioms for the GM_H, and Axiom H. It should be noted that Axiom H, first proposed by Woodin, is not part of the axioms of the GM_H proposed by Steel, so is not integral to the GM_H. But, since it proves that the generic multiverse has a core, I will always include it (thus, when speaking of the the GM_H, I will always intend the "generic multiverse + Axiom H").

To characterize the GM_H in a more formal way, we need a multiverse language and an open-ended multiverse theory. The *multiverse language* MV consists of the usual syntax of set theory, but with two sorts instead of one: one for the *sets* and one for the *worlds*. It is a sublanguage of the standard one, and it is expressive enough to state versions of the axioms of ZFC and large cardinal hypotheses[21], preserved by set forcing (this means that, when applying forcing, the truth value of large cardinals hypotheses is preserved, see section 6.2.1 for some details on forcing):

Axiom 11 (Axioms of MV) *1. For each axiom φ of set theory and for every world W of the multiverse, there exists a translation of φ in W, denoted φ^W;*[22]

[21] A *large cardinal hypothesis* is an axiom that states the existence of large cardinals, very large transfinite cardinal numbers (roughly, numbers so big that they cannot be reached from below with the usual set theoretic operations).

[22] So this is actually a schema for every ZFC axiom.

2. *Every world is a transitive proper class. An object is a set just in case it belongs to some world;*

3. *If W is a world and $\mathbb{P} \in W$ is a poset, then there is a world of the form $W[G]$, where G is $\mathbb{P}$-generic over W*[23]*;*

4. *If U is a world, and $U = W[G]$, where G is $\mathbb{P}$-generic over W, then W is a world;*

Axiom (1) controls the translation between the language of the multiverse and ZFC, while axioms (2)-(4) control which models are actually universes of the multiverse and also they define how to produce them. In particular, this axioms explains why this is a *generic* multiverse: the reason is that we use set-generic forcing to produce the universes of the multiverse. The following axiom is also about the construction of new universes and their relations between them:

Axiom 12 (Amalgamation) *If U and W are worlds, then there are G and H set generic over them such that $W[G] = U[H]$.*

This says that, given any two world, we can use forcing to extend them in such a way the extensions are actually the same universe. We can construct a model of this multiverse (by forcing[24]). Also, everything we can say in the multiverse language can be expressed using just *one* world-quantifier. Moreover, we can introduce to the multiverse the large cardinal hypotheses that are preserved by small forcing.

Our mathematical knowledge of the GM_H is still very limited, but Woodin and Steel have manage to prove some basic facts. The main tool they assumed is the following axiom:

Axiom 13 (Axiom H) *For any sentence φ in the language of set theory (LST): if φ is true, then for some $M \models AD^{+}$*[25] *$+ V = L(P(\mathbb{R}))$*[26] *such that $\mathbb{R} \cup OD$*[27] *$\subseteq M$, $(HOD$*[28] *$\cap V_\Theta)^M \models \varphi$.*

This axiom says that V looks like HOD^M. That is, it roughly means that every member of V can defined in a model M in terms of a finite number of ordinals by a first order formula. We can prove the following facts about this axiom:

Fact 1 *Axiom H implies the CH and most instances of the GCH.*

Fact 2 *If M is a model of $AD_{\mathbb{R}}$ plus "Θ is regular", then Axiom H holds in $(HOD \cap V_\Theta)^M$. Consequently, it is consistent with weak large cardinal hypothesis.*

But the main consequence of Axiom H is that it implies that the multiverse has a core and that this core satisfies Axiom H.

In summary, we can say that Axiom H has the following advantages:

1. it implies that the multiverse has a core;

2. it suggests an approach to develop a *fine structure theory* of this core;

3. it may be consistent with all the large cardinal hypotheses (although this is still an open problem)

The fact that Axiom H implies that the multiverse has a core can hardly be overstimated. The core guarantees the foundational character of the multiverse. Indeed, since, as proved by Woodin, the core of the multiverse is ZFC, Axiom H effectively guarantees that the GM_H can be used to model all the usual mathematics, thus preserving all mathematical practice.

[23] A theorem by Laver and Woodin (Woodin (2011)) states that there is formula ψ of LST such that if $N = W[H]$ (where H is N-generic for a set forcing $\mathbb{P}$) then ψ defines W over N from $\mathbb{P}$ and H. We can use this result to precisely formalize this axiom, defining W via ψ inside U.

[24] For an informal description of forcing, see section 6.2.1.

[25] AD^+ is a strengthening proposed by Woodin of the axiom of determinacy AD. It is used here for purely technical reasons that are outside the scope of this disposition.

[26] $L(P(\mathbb{R}))$ is the smallest inner model that contains the power set of $\mathbb{R}$.

[27] A *ordinal definable* set, in other words a set definable over the universe of sets from ordinal parameters.

[28] A set is *hereditarily ordinal definable* if and only if it and all members of its transitive closure are OD.

The possibility to develop a fine structure theory[29] is also very important. It is needed to study in-depth the definability of hierarchies: historically, it was developed to study L, and is therefore particularly suited for the study of constructable hierarchies. Moreover, from a mere technical point of view, fine structure theory is necessary when building a special type of models: *core* models. Since it follows from Axiom H that our multiverse has a core, it is very useful to have an instrument to better build such a core and to better investigate the hierarchies stemming from that model.

Given how little we know about the GM_H, this project aims to answer the following questions:

Main research goal Develop the GM_H from a philosophical and mathematical point of view, arguing that is the most suited multiverse conception to replace V as the new set theoretic universe.

Research Goal Definition of an appropriate concept of mathematical truth for the GM_H.

Connected to this more philosophical goal, the following open questions should be addressed, to deepen our mathematical knowledge of the multiverse:

Question 1 *Which core is the more appropriate for our endeavor? Is ZFC the only viable option or could we choose a weaker theory (like ZF^-), thus maximizing the possible universes that are part of the GM_H? On a related note, what are the consequences of changing the core?*

Question 2 *What could be a successful formalization of the informal axioms proposed by Steel (2014)?*

Question 3 *What is the consistency strength of the consequent formalized multiverse theory?*

5.1 A naturalistic justification of the GM_H

As stated in section 1, one of our aims is to justify the GM_H from a naturalist point of view, thus ensuring that it satisfies both desiderata 4 and 5. Generally speaking, this means considering mathematics itself as the final judge about the possibility of replacing the Single Universe V with the GM_H as a foundational theory for mathematics. I will argue that the GM_H is in fact better than the Single Universe, and should be considered the best framework for mathematical practice.

To this end, I will use the two naturalistic principles introduced by Maddy, MAXIMIZE and UNIFY. My argument can be divided in two parts:

1. first, I will prove that the GM_H is just as foundational than the Single Universe (desideratum 4, UNIFY);

2. second, I will show that it proves more isomorphisms than the Single Universe (desideratum 5, MAXIMIZE).

The aim of this argument is to show that the GM_H scores better than the Single Universe on both desiderata 4 and 5 (Maddy's naturalism criteria), thus providing a better framework for mathematical practice.

In order to be a foundational theory just like the Single Universe, the GM_H should provide Meta-mathematical Corral, Elucidation, a Shared Standard for proofs and Risk Assessment. ZFC, and consequently the Single Universe, is very precise in delivering all those features: we can prove something general about all mathematics in it, since it can all be reduced in one "manageable package" (Meta-mathematical Corral); we can use set-theoretic definitions to clear otherwise ambiguous mathematical objects (Elucidation); it is a very good standard for what counts as a mathematical proof (Shared Standard); and, finally, it is very useful when trying to assess the problems with the use of a certain mathematical object or method. All in all, we can say that ZFC does a very good job as a "foundation" for mathematics.

How can the GM_H be just as good? We have already an example of a rival theory to ZFC, category theory, that tries to replace it as the foundational framework for mathematics. But it

[29]Without getting too technical, a *fine structure* is a method to "stratify" hierarchies (like the cumulative hierarchy) to better study them.

fails: indeed, it ends up trying to emulate set theory, but not as good as ZFC (in particular, it is worse as a Shared Standard and for Meta-mathematical Corral). So, how can the GM_H compete with ZFC and the Single Universe, that seem to be tailored for the foundational role?

The solution is the existence of a core for the generic multiverse. As we have already seen, the multiverse core is a set of truths common to *every* universe of the multiverse. This core should stabilize our multiverse, avoiding universes too wild or too weak to do the foundational job. The core assures us that the rules are the same in every universe, so our definition of proof will be preserved across the whole multiverse (Shared Standard). Also, the language and rules we are using will be the same in all the multiverse, so a definition of a mathematical object in one universe will still be understandable in other universes (Elucidation). Moreover, if a mathematical object clashes with something in our core it will be problematic to use it in every universe of the multiverse (Risk Assessment). Finally, we will be able to prove something general about not only mathematics, but also the multiverse itself, since it will be included in the core (Meta-mathematical Corral).

Now we have two competing frameworks: the Single Universe and the GM_H. Both fare equally good regarding UNIFY. Still, the GM_H has no edge against the Single Universe. Indeed, it requires a terrible mathematical machinery (the language MV, the new axioms, axiom H) to be *just as good as* the Single Universe. So, if the situation would stay like this, obviously trading the Single Universe for the GM_H would not be the best course of action.

To further study the matter, we can compare the two framework in respect to the range of available isomorphisms types provable in them (MAXIMIZE). This principle states that, since the aim of set theory is to represent all the known mathematics within a single theory, it should *maximize* the range of available isomorphism types. This is particular important for mathematics since, through the isomorphism, it is possible to "import" methods and results from a mathematical field to another. I argue that the classic set theory ZFC is *restrictive* over the GM_H, that is, the GM_H *strongly maximizes* over ZFC in the sense that it provides a wide range of isomorphism types that are not available in ZFC. The GM_H strongly maximizes over ZFC: there is no theory T extending ZFC that properly maximizes over the GM_H and the GM_H inconsistently maximizes over ZFC. This means that the GM_H provides structures that cannot be satisfied by ZFC, even if properly extended. To see this, assume that the core of the GM_H is ZF^- (set theory minus the Axiom of Foundation). From this core we can build a multiverse in which, among others, there is a universe for ZFC and a universe for $ZF + AD$. In this multiverse I can have *both* the Axiom of Choice (provided by ZFC) *and* a full Axiom of Determinacy (provided by $ZF + AD$). Determinacy and Choice are actually incompatible, but they can coexist in the GM_H. Hence, the GM_H, unlike the intended model of ZFC, can include all the structures based on Determinacy. That is, the GM_H provides a new isomorphism type, that is, it proves the existence of a structure that is not isomorphic to anything in ZFC.

Furthermore, the GM_H also provides what Maddy calls a *fair interpretation* of ZFC, i.e. the GM_H validates all the axioms of ZFC (this is because ZFC is part of the multiverse) and one can build natural models, inner models, and truncations of proper class of inner models at inaccessible levels of ZFC in the GM_H.

Thus, we can conclude that the GM_H satisfies both naturalistic principles: while is just as good as the Single Universe regarding its foundationality, it is a better Generous Arena for mathematical practice.

5.2 Further developments of the GM_H

Steel's construction of the GM_H is very compelling and, as argued both by Maddy (2017) and Koellner (2013), is probably the most promising multiverse conception. However, it still has some problems. To begin with, Steel doesn't actually provide an explicit formalization of MV's axioms.. While this is not an important problem, the construction is clear and precise even without this formalization, having a formalized axiomatic system is the required next step. Second, the CH and the large cardinals hypotheses are absolute across the whole multiverse. This means that, even though they are not part of the core, there is no universe of the multiverse in which they are false. One of my goals of my project is to improve the GMH on both of these counts.

One of my goal is to provide a formalization of the axioms of the GM_H. This explicit axiomatization will be a two sorted first order theory, very similar to the MK class theory. Moreover, with this formal characterization of the axioms will be possible to use the notion of restrictiveness as defined by Maddy and Löwe[30] to prove various results regarding MAXIMIZE.

Once we have done that, we can further investigate some mathematical detail of the GM_H. In particular, I plan to address the problem of absoluteness arising from Steel's formulation. In Steel's view, the multiverse is build by applying set - generic forcing to ZFC. In this way, large cardinals hypotheses and CH are *absolute* across all the multiverse. This doesn't mean that they are part of the core: there are universes in which they are still indeterminate, e.g. in the core CH is indeterminate, or in the universe ZFC + "there exists an inaccessible cardinal" the existence of a Mahlo cardinal is indeterminate. But this is a problem: first, it is arbitrary to choose which statements are absolute in the multiverse (e.g., why not $\neg CH$? or $V = L$). Moreover, this is not in line with mathematical practice, for which there is no consensus on the truth of these statements. Thus, we need to be open as much as possible, otherwise we will fall in the same problems of universism.

While this is the intended behaviour that Steel aims for, I propose a different approach, to ensure that both CH and large cardinal hypotheses are not absolute. First, instead of choosing ZFC as the multiverse core, I argue that a much weaker core like ZF^{-}[31] is more suited, since it opens many more possibilities (e.g. in this case all the universes not well-founded are part of the multiverse). Moreover, instead of using set generic forcing, I propose to use *class*-generic forcing. This kind of forcing is applied not adding a new set to a transitive model, but adding a new class. With this method, large cardinal hypotheses and CH are no longer absolute across the whole multiverse, and thus the multiverse is much more open ended and respectful of mathematical practice.

6 Methodology

6.1 Logical analysis and instruments

To analyze the problem in the most thoroughly way we have to make use mainly of logical methods. In fact, relying just on philosophical abstract reflections would not be enough, since quite a few details would be lost without a more formal method of analysis. The natural choice is, obviously, mathematical logic.

But here a doubt arises: which logic is better suited for our endeavors? There are mainly two options: classical first order logic or an intuitionistic/constructive logic. The latter would be very helpful in sorting out which sets are part of the multiverse and which not. However, would be very problematic to pursue our goals in an intuitionistic framework. In fact, while they are not as strong as classical logic, they are still quite strong and, moreover, they are both used to develop probably the most important non classical set theories: intuitionistic set theory (IZF) and constructive set theory (CZF). Both theories are well-known and worth researching, but none of them are suited for the generic multiverse. In fact, since we still cannot add very large cardinals to these theories (they can only model so called measurable, i.e. smaller, cardinals), the adoption of either logic would be in tension with the generic multiverse and, more specifically, with its underlying *pluralistic* conception.

On the other hand, classical first order logic is the most obvious and natural choice. First of all is the strongest logic with both the compactness theorem and the Löwenheim - Skolem theorem (as proved by Lindström)[32]. Moreover, it is the logic on which all mathematical practice is actually based upon. Thus, from a naturalistic point of view, there is no other choice than to use classical first order logic.

In conclusion, since we cannot avoid using model theoretic instruments and arguments, and since we need a framework that is as strong as possible, it is arguable that first order classical logic is the only possible choice.

[30] See Maddy (1998).

[31] The same axioms of ZFC, but without the AC and the Axiom of Foundation.

[32] More precisely, Lindström theorem says that if a non classical logic has both the compactness theorem and the Löwenheim - Skolem theorem, then is weaker than classical first order logic.

6.2 Set Theoretic Methods

Since this project is at the interface between mathematics and philosophy, we will rely heavily on mathematical (set theoretic) methods. For one thing, in keeping with naturalism about set theory, the arguments put forward in the course of this project will have to be acceptable by mathematical standards. For another, since we are dealing with the multiverse, we cannot restrict ourselves to just describe it, but we need some instruments to actually produce it. These instruments are borrowed from set theory, and they are forcing (section 6.2.1) and inner models (section 6.2.2). In the next sections, I briefly introduce them in turn.

6.2.1 Forcing

Forcing is probably the most important instrument developed in set theory in the last few decades. First introduced by Cohen to prove the independence of CH and AC from ZFC, now is mainly used to produce models and consistency results. It is in effect a very useful method, especially in large cardinal theory and in proving relative consistency results. To use this method to build a multiverse, we have to assume at least a very simple form of realism, i.e. we have to take mathematical theories at face value: if a mathematical theory entails the existence of a certain object, then that object exists. For instance, numbers exist simply because Peano Arithmetic proves that numbers exist. In our case, since the application of forcing says that a certain model exists, we should come to the conclusion that that particular model exists. This should not be a problem, since it is in line with our main philosophical assumption, naturalism.

The main idea of forcing is to extend a transitive model M of set theory (a *ground model*) by adding a new set G (a *generic set*) to it. This generic set is approximated in the ground model by *forcing conditions*, that determine what is true in the extension. In this way we obtain a larger transitive model of set theory, $M[G]$, called a *generic extension*.

Cohen original method was to start with a countable transitive model M of ZFC and a set of forcing conditions in M. From this, it is easy to prove that a generic set exists, and the main result was to prove that $M[G]$ was a model of ZFC and that the CH fails in $M[G]$.

But there is a main disadvantage in this method: if we apply this set generic forcing to the multiverse core ZFC we have the consequence that large cardinal hypotheses and CH are *absolute* across the multiverse. This means that, while they are not part of the core – so there are universes in which they are indeterminate – there are nevertheless no universes in which they are false! While this is the intended behaviour that Steel aims for, I propose a different approach, to ensure that both CH and large cardinal hypotheses are not absolute. First, instead of choosing ZFC as the multiverse core, I argue that a much weaker core like ZF^-[33] is more suited, since it opens many more possibilities (e.g. in this case all the universes not well-founded are part of the multiverse). Moreover, instead of using set generic forcing, I propose to use *class*-generic forcing. This kind of forcing is applied not adding a new set to a transitive model, but adding a new class. With this method, large cardinal hypotheses and CH are no longer absolute across the whole multiverse, and thus the multiverse is much more open ended and respectful of mathematical practice.

6.2.2 Inner Models

Inner models are the counterpart of forcing. While forcing allows for the construction of various set theoretic models from a given large cardinal, inner models offer a way to analyze the complexity of those models in terms of large cardinals. This analysis is fundamental, since it is used to determine lower bounds for the consistency strength of the set theoretic principles we are testing.

Historically, the inner models can be traced back to Gödel's study of the fine structure of the constructible universe L. The first explicit use of this method originated in Jensen's analysis of the constructible universe, and was later developed in the study of the core model by Dodd and Jensen. After that, it was mainly pursued by Jensen, Mitchell, and Steel, among others.

Usually, the main goal of inner models theory is defining a *core model* K which can coexist with larger cardinals in V and with the following properties:

[33] The same axioms of ZFC, but without the AC and the Axiom of Foundation.

1. is like L;[34]
2. it is close to V.[35]

However, this approach is not optimal for our goal. First of all, the core model is *canonical*, that is, it cannot be changed by forcing. But, since our project is rooted in the multiverse, we need something that could be changed by forcing (remember that forcing is the first instrument to create new universes). Also, we want to be able to work with very large cardinals that are unknown or poorly understood with canonical inner models. Moreover, since we are dealing with a multiverse, it is possible that some universes will be models of $ZF + \neg CH$, or $ZF + \neg AC$. But canonical inner models cannot handle these cases very well.

It should be noted that there are no problems in considering a core model that is like L and close to V. For instance, while we do not accept $V = L$[36], we can still have L. Also, we can still satisfy the second property (that the core model should be very close to V): in fact, V looks like the HOD of a model of AD (for the technical detail, see Steel, 2014), and we want that our core to be very similar to V.

For these reasons, a better approach is the class HOD of hereditarily definable sets and its variants. In effect, the main feature of HOD that is always cited as a disadvantage, the fact that it is not canonical, is actually what we are looking for. Moreover, it is very well suited when dealing with models where the axiom of choice fails (hopefully not our case) and with determinacy. Finally, it is perfect for very large cardinals and it is deeply connected to Axiom H.

7 Conclusions

Which conclusion can we deduct from all of the above? First of all, that classic set theory as it stands – the cumulative hierarchy generated from ZFC – it is not enough, and has to be extended in some way. While this is hardly a surprise – we know this from the sixties, when the independence results were first proven – how we actually carry out this extension matters, and there are very different approaches to do it. A common approach considers the concept of set *currently* indeterminate, but, in the future, the hope is that we manage to "find out" which extension is the right one, thus determining *the* concept of set. On the other hand, we might draw the conclusion that there is no *single* concept of set, and we carry out mathematics in what is essentially a multiverse. This is not to say that we don't have any intuition on what a set is, or on how the membership relation works: this intuition, exemplified by the cumulative hierarchy, is still the right one. Simply put, it can account only for the starting point, and then we have several way to extend it. The usual objections against this conception, categoricity and the possibility to simulate different models in V itself, can be dispatched. Categoricity is more a consequence of the logical features of the theory, than the theory itself, thus the determinacy that arises in ZFC_2 it is not a consequences of the set theoretic axioms, but of the fact that it is a second order theory. On the other hand, the possibility to simulate non standard models in the intended model is very limited: we can simulate only a models at a time, and moreover we are not gaining any insight on which is the "intended" model (i.e. we can take a non-standard model and simulate in it the "standard" one). Another argument against the multiverse and in favour of the classic single universe is Maddy's naturalism: if we accept her assumptions and the principles UNIFY and MAXIMIZE, we are bounded to deduce that the single universe is better (that is, more naturalistic) than the multiverse. As I have shown in the previous sections, we can reverse are argument and make a very strong point that the multiverse is actually more naturalistic than the single universe: if we build the right kind of the multiverse, we can have a multiverse that is foundational – that is, it satisfies UNIFY – and it maximizes the isomorphisms types available.

Thus our endeavor should be directed to find out which characteristic this multiverse should have to qualify as the new "foundation" of mathematics. The following are our *desiderata*:

1. the multiverse should encompass our intuitive notion of set and membership;

[34] This property is satisfied by defining it as one if the models $L[\mathcal{U}]$ or $L[\mathcal{E}]$.

[35] This property can be satisfied with a *covering lemma*, but it changes from one case to the other (for example, when L is the core model, then we can use Jensen's covering lemma).

[36] Basically, because it is restrictive on the possible isomorphisms. For a detailed criticism, see Maddy, 1997.

2. the multiverse should come with an axiomatic theory of it;
3. this axiomatization should be very high in the consistency strength hierarchy;
4. the multiverse should be foundational (UNIFY);
5. the multiverse should prove as much isomorphisms types as possible (MAXIMIZE);
6. the multiverse should be as open ended as possible, including all the possible universes.

Since there are several multiverse conceptions, we can follow them to argue against some of them, to pinpoint the multiverse conception which satisfy them more closely. Hamkins' broad multiverse doesn't save our intuitive notion of set and membership, and it is not foundational in any sense. Also, it cannot be properly formalized. All in all, while it fares very well on the last two desiderata, it has some inherent problems that seem too serious to be cope with. On the other hand Woodin's generic multiverse GM_{Ω} has axioms, has a very high consistency strength and, in a very loose way, could be considered foundational. But it is build upon very strong mathematical hypotheses – namely the Ω Conjecture – that limits our ability to carry out mathematics, thus failing in providing us the last two desiderata. It also has problems in encompassing our intuitive notion of set and membership. Väänänem's multiverse is still rather underdeveloped, and while it has an axiomatization, and respects our intuitive notion of set and membership, it is not foundational, and it is even worst than the single universe when considering MAXIMIZE. This is because in Väänänen's multiverse all the universes are hidden from each other: since we cannot access them in any way nor talk about them – his axiomatization doesn't have any variable for the universes – we cannot prove any isomorphisms between them, and we cannot even know which universes are part of the multiverse and which not. Finally, Friedman's Hyperuniverse is also problematic: since it considers only the countable transitive models of set theory it doesn't agree with our intuitive notion of set, it doesn't have a clear axiomatization and cannot be considered foundational.

The last multiverse conception taken into account is Steel's generic multiverse GM_H. Among all the multiverse conceptions is the one that agrees with our intuitive notion of set and membership the most. For these reasons could be considered foundational in the Maddy's sense. Moreover, it proves several isomorphisms types more than the single universe. Consequently, we can argue that the GM_H is the multiverse conception that satisfies all the desiderata more closely. But it could be better, especially for the last two desiderata – for example, Hamkins' broad multiverse is better in dealing with them. The problem here lies in how Steel defines his multiverse: from ZFC, the multiverse core, he uses set-generic forcing to generate the multiverse. In this way, for example, large cardinals are absolute across all the multiverse. This restrict our possibilities, and restrict the overall power of the multiverse. A possible solution to this problem is to consider a much weaker core (e.g. ZF^-) and class - generic forcing instead of set - generic. This class - generic multiverse is even more powerful, without losing any of the foundational characteristics of the set - generic one.

In conclusion, in this disposition I have argued that classical set theory it is not enough to be a foundational framework for mathematics, and thus has to be extended in some way (2). A multiverse conception of set theory is a recent account developed to do that (4.2.1). I have argued that it is safe from categoricity arguments (4.1.2), but also that most of them don't work (4.3). Finally, in section 5 I have presented my main point, i.e. that Steel's GM_H is the multiverse conception that more closely satisfies all our desiderata for a foundational framework for mathematics.

7.1 Timeline

The first six months will be devoted to a preliminary literature research; the second half of first year and the second year will be devoted to the actual research on the specific aims of the project; finally, the third year will be devoted to writing and reviewing the dissertation.

Going into some more details. The main objective of the first half a year is to carry out a preliminary bibliographical research on the background and the specific aims of the project. To gather more feedback, I will plan to discuss the project (in particular the aims, the methodology and the possible consequences) at the end of the first year (roughly around September 2019).

During the second half of the first year and the second year I will focus on the research around the specific aims of the project. After some time spent on bibliographical research, I will start analyzing the main object of the project. Towards this end, a visiting period, during the second year at UC - Irvine will be of the outmost importance for the outcome of the project. At UC - Irvine, I will be able to discuss the ongoing project with Professor Walsh (an expert on pluralism) and Professor Maddy (an expert on naturalism in mathematics). This should allow me to polish both my philosophical claims (with the help of Professor Maddy's expertise) and the more technical details (with Professor Walsh). Also, having the possibility to visit an institution so different from Salzburg University will surely help me in preparing for my future career. For all these reasons, the period should be a prolonged one and I plan to stay there for a full academic year. This will permit me to get the most out of it.

The last year is dedicated entirely to the dissertation, which I am planning to submit by September 2021.

I can summarize this timeline as follows:

- **September 2017 - February 2018**: preliminary literature research;
  - **September 2017 - November 2017**: consolidating the philosophical assumptions of the project;
  - **December 2017 - February 2018**: analyzing the mathematical background needed for the specific aims;
- **March 2018 - May 2020**: research on the specific aims of the project;
  - **March 2018 - September 2018**: working on the mathematical characterization of the multiverse;
  - **September 2018**: exposition of the preliminary results at FFNC 2018;
  - **October 2018 - May 2019**: visiting period at UC Irvine, integration of pluralism with naturalism;
  - **June 2019 - May 2020**: final assessing of the project goals and exposition in international conferences;
- **June 2020 - April 2021**: preparation of the Dissertation;
- **May 2021 - August 2021**: final review;
- **September 2021**: Submission and discussion of the Dissertation.

7.2 Table of contents

The following is a first sketch of the Dissertation's table of contents:

Chapter 1 : Introduction

1.1 Preliminaries on the Philosophies of Mathematics involved

1.1.1 Naturalism

1.1.2 Pluralism

1.1.3 Anti-Pluralism

1.2 The historical reasons of the emergence of the Multiverse

1.2.1 Independent propositions

1.2.2 Alternative set theories

1.3 Structure of the Dissertation

Chapter 2 : A Second Philosophy of Mathematics

2.1 : UNIFY

2.1.1 Meta-mathematical Corral

Symbols and acronyms used

CH : Continuum Hypothesis;

V : The universe of set theory;

ZFC : Zermelo-Fraenkel with Choice;

ZF : Zermelo-Fraenkel;

ZF$^-$: Zermelo-Fraenkel minus Foundation;

GCH : Generalized Continuum Hypothesis;

MV : The multiverse language formalized by Steel;

$\mathbf{GM_H}$: The Generic Multiverse + Axiom H;

LST : Language of Set Theory;

AD : Axiom of Determinacy;

L : Constructible universe of set theory;

AC : Axiom of Choice;

HOD : Hereditarily ordinal definable.

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