Against Categoricity

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Abstract

According to the universism position in set theory, there is only a Single Universe for set theory and all the models we build through forcing have instrumental value: they are useful only until we find the "true" one. The belief that this universe actually exists and the consequent dismissal of a pluralist conception is based on two main arguments: first, already in V we can simulate different universes, so there is no need to assume the existence of other universes: second, set theory is actually categorical, i.e. all the different models stemming from it are in fact isomorphic. I argue that both arguments are off target.

1 Introduction

From its beginning, classic set theory has been quite the success story: it is an axiomatic theory in which all the known mathematics can be expressed. It was later discovered that its classical axiomatization, ZFC, was not enough to settle key questions such as the Continuum Hypothesis (CH), that are independent from the axioms of set theory. Since then, the quest for new axioms to settle these independent questions began. This, in addition to the development of forcing, gave rise to a plurality of set theoretic theories and models: a multiverse.

The advocate of universism argues that all these theories and models have only instrumental value, and that somewhere down the line we will be able to decide which theory is the "right" set theory. This argument takes strength from categoricity results and the possibility to simulate non standard models of ZFC inside V itself. I argue that these two arguments are more problematic than it is usually thought, and that they don't pose a threat to the pluralistic conception of set theory.

A theory T is categorical if and only if all of its model are isomorphic, which is essentially to say that T has only one model (up to isomorphism). It is well known that no first-order axiomatisation of sets, such as ZFC, may provide us with just *one* isomorphism type corresponding to the "structure of all sets". This is because first-order theories admit of a huge number of non-isomorphic models, which makes it impossible for them to meet the modelist's desideratum. On the other hand, *full* second-order set theory (such as, ZFC_2, that is, ZFC with second-order versions of the Separation and Replacement Axioms), may, in principle, fit the bill, thanks to Zermelo's quasi-categoricity theorem for ZFC_2.

Zermelo proved that, given any two models M and N of ZFC_2, either $M = N$, or M is a rank initial segment of N, that is, M and N must have the same *width*, but they might differ in *height* (and the height must, at least, be that of V_κ, where κ is a *strongly inaccessible cardinal*).[1,2] So, the universist might thrive on the quasi-categoricity result by asserting that any sufficiently strong second-order theory of sets (such as ZFC_2) will provide us with a good approximation of the relevant isomorphism type of the 'intended' structure of all sets. ZFC_2 settles most of the independent propositions (in particular the CH).

However, this second-order determinacy is seen by most set-theorists simply as a consequence of the in-built *logical* features of the theory, rather than of the second-order axioms' ability to express a more determinate concept of set. Thus, set-theoretic indeterminacy has progressively come to be viewed as an integral and stable feature of the current set-theoretic landscape, and most of present set theoretic is carried out within (first-order) ZFC and, indeed, systematically exploits its radical indeterminacy.

2 Against universism

As mentioned, universism is the thesis that there is only one set theoretic universe, V (the canonical universe of set theory). This universe is the so called "canonical" model of set theory, as opposed to all the others models, the *non-standard* models. For example, the constructible universe L is a non-standard model of set theory.

Although it is true that set theorists make use all kinds of non standard models of ZFC, universists typically insist that each of these models can

[1] In Zermelo (1930).

[2] A cardinal κ is said to be inaccessible if: (1) it is regular and (2) limit. The least inaccessible $> \aleph_0$ cannot be proved to exist in ZFC. κ is *strongly inaccessible* if it is inaccessible and, for all $\lambda < \kappa$, $2^\lambda < \kappa$.

be simulated within V. An argument against pluralism is that in the single universe V we can actually simulate any non standard model of set theory. For instance, in ZFC we can simulate a model of $ZFC + V = L$ or a model of $ZFC + LCs$ (i.e. $ZFC+$ Large Cardinals axioms), even though they are incompatible.[3] However, we cannot simulate them *at the same time*. This means that in V we can have a simulation of $ZFC + V = L$ in the canonical model, but then, when trying to simulate $ZFC + LCs$ we are forced to throw away everything that was proved in the simulation of $ZFC + V = L$. The main consequence of this fact is that we cannot compare two non standard models at the same time. By contrast, all the different models are available in the set theoretic multiverse, at the same time, and we can prove isomorphisms between their structures. Thus, compared with a pluralistic conception of set theory, we are loosing the ability to investigate those models synchronically. This means that, when comparing a single universe prospective with a multiverse prospective using the principle MAXIMIZE[4], the latter will fare better than the former. Consequently, from a naturalistic point of view, a multiverse conception of set theory is a more apt to include and compare any mathematical object (this is the Generous Arena role from Maddy, 2017) than the Single Universe.

3 Against categoricity

The multiversist might bite the bullet and insist that the study of isomorphisms between non standard models of set theory isn't the proper subject of set theory, and that when set theory is properly axiomatised, at the second-order, it has only one model or, at least, it is quasi-categorical. More specifically, the argument assumes that set theory must be formulated at the *second order*. That is, set theory must not only quantify on *members* of the domain, but also on *arbitrary subsets* of the domain.

Now, the significance of Zermelo's theorem is highly disputed.[5] As has been frequently pointed out, the desired philosophical consequences of the

[3] $V = L$ is the Axiom of Constructability, that says that all the sets of the universe can be build from simpler sets, and it is incompatible with the existence of most large cardinals (LCs).

[4] According to this principle, when comparing two theories the one that can prove more isomorphisms types is preferable, see Maddy, 1997.

[5] Zermelo himself does not encourage this interpretation of his theorem in Zermelo (1930) (in Ewald (1996), pp. 1232-33), but the point is controversial. Other authors have construed the quasi-categoricity of ZFC_2 as rather supporting: (1) the existence of a single concept of set (Martin (2001)), or (2) the truth-value determinacy of all set-theoretic statements (McGee (1997)).

theorem are available only if one adopts the full semantics of second-order logic, while an alternative semantic approach, *Henkin semantics*, not conducive to the quasi-categoricity result, is always possible. Moreover, Meadows (2013)) makes the more general claim that, while categoricity proofs (including Zermelo's proof) may successfully prove that, under certain conditions, given a realm of mathematical objects, there is certainly one theory which picks up a unique structure of them, they cannot help establish that there is one natural, intuitively given unique structure which incorporates all the properties of that realm.[6]

Button and Walsh, elaborating upon McGee (1997), Parsons (1990) and, partly, Putnam (1979), have come up with a reformulation of this argument which relies, instead, upon an *internal* version of categoricity. Internal categoricity is, very roughly, the idea that categoricity may be formulated, within a suitable second-order logical framework, in a purely *deductive* way, without having to deal with and/or endorse *semantic* facts whose philosophical value is controversial. In particular, supporters of internal categoricity will place great emphasis on what McGee calls the "second-order intolerance to truth-value indeterminacy", that is the fact that, for all φ expressed in the language of second-order set theory, it is possible to prove that either φ or $\neg\varphi$ is a theorem, and none on Zermelo's quasi-categoricity theorem.

McGee and Martin have tried to revamp this argument, arguing that determinacy it is not a feature of the logic used to formulate the theory, but a consequence of the axioms itself. To this end, McGee (1997) proves that second order set theory with urelements[7] ($ZFCU_2$) is also categorical, while Martin (2001) argues that the notions of set and membership are unique and thus can be represented only by a categorical axiomatization.

The problem with both of these arguments is that they both rely on extremely strong assumptions: McGee's theorem is based upon the Urelements Axiom (that says that there exist urelements), while Martin arguments are based on the Uniqueness Postulate (a form of extensionality that applies not only in a structure, but *across all structures*). It can be argued that both assumptions have an ad hoc character: while they help proving categoricity results, they are not really used by mathematicians. More precisely, they are not needed to prove standard mathematical results.[8]

More generally, the main problem with categoricity results is that the concept of set is an *algebraic concept*. Briefly, this means that set theory

[6]As the latter belief already implies that either the concept of those objects is unique, or that any further underpinning of the properties derivable from that concept is unique.

[7]An urelement is an object that it is not a set, but could be a member of a set.

[8]In particular, Martin's Uniqueness Postulate is exceedingly strong: it allows one to prove that even *first-order* set theory is categorical!

is no different from group theory or any other algebraic theory. The main intention when formulating these theories is to admit non isomorphic models, since mathematicians need different models for different purposes[9]. All these models are considered totally legitimate by the majority of working mathematicians. The main consequence of this attitude is that second-order determinacy is seen by most set-theorists simply as a consequence of the in-built *logical* features of the theory, rather than as a consequence of the second-order axioms' ability to express a more determinate concept of set. Thus, set-theoretic indeterminacy has progressively come to be viewed as an integral part and stable feature of the current set-theoretic landscape.[10]

4 Conclusion

In conclusion, universism faces serious difficulties. The main argument usually considered in its defense, categoricity, is quite problematic, and doesn't give a satisfactory solution to the foundational problems in mathematics and set theory.

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[9]For example, consider the hierarchy of large cardinals, as superbly described in Kanamori (2003).

[10]As argued by Mostowski (1967) and, more recently, Hamkins (2012).

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