Against universism

Universism is the thesis that there is only one set theoretic universe, V (the canonical universe of set theory). This universe is the so called "canonical" model of set theory, as opposed to all the others models, the *non-standard* models. For example, the constructible universe L is a non-standard model of set theory. Although it is true that set theorists make use all kinds of non standard models of ZFC, universists typically insist that each of these models can be "simulated" within V and that, in the end, they are only "simulated universes". Thus, universists typically argue against pluralists about set theory on the ground that the non-standard universes that populate so-called multiverse conceptions of set can be simulated within ZFC. For instance, in ZFC we can simulate a model of $ZFC+V=L$ or a model of $ZFC+LCs$ (i.e. $ZFC+$ Large Cardinals axioms), even though these two models are incompatible.[1] In this paper, I argue against this universist strategy is fundamentally problematic. The main problem is that, although in the single universe V, we can actually simulate any non-standard model of set theory, we cannot simulate them *at the same time*. This means that in V we can have a simulation of $ZFC+V=L$ in the canonical model, but then, from within V we cannot simulate $ZFC+LCs$ we are forced to throw away everything that was proved in the simulation of $ZFC+V=L$. The main consequence of this fact is that we cannot compare two non standard models at the same time. By contrast, all the different models are available in the set theoretic multiverse, at the same time, and we can prove isomorphisms between their structures. Thus, compared with a pluralistic conception of set theory, the universist conception loses the ability to simulate these models synchronically. This means that, when comparing a single universe prospective with a multiverse prospective using, for instance, the Maddy's naturalistic principle MAXIMIZE[2], the latter will fare better than the former. Consequently, from a naturalistic point of view, a multiverse conception of set theory is, in Maddy, 2017 terminology, a more *generous arena* than the Single Universe. In this paper, I argue against universism, defending instead a pluralistic conception of set theory that admits that non standard models are something more than simple simulations[3].

I first need to clarify what a "simulation" is. In set theory, we have the classical axiomatization ZFC and its canonical model, the cumulative hierarchy V. A non standard model of ZFC is a model produced from ZFC and V through the application of set generic forcing. With forcing, we can "create" a new model of ZFC: the usual example is the mutually incompatible models $ZFC+CH$ and $ZFC+\neg CH$. In this case, we are creating two new models, V' and V^*, in which the Continuum Hypothesis is, respectively, true and false. These two models appear to be "fatter", i.e. larger than the original V: they are usually considered *width extensions* of V, produced by the addition of new subsets to the cumulative hierarchy.[4] However, this set forcing cannot be applied to the whole V, but only to countable sets. Consequently, what is actually going on with forcing, is that we are taking a countable set *in* V that "simulates" the whole universe, we then apply set forcing to it to produce its width extension, and thus produce a model of, for example, $ZFC+CH$. But since we started with a countable set *inside* V, we are not producing a whole new universe, but only a slighter larger countable set inside

[1] $V=L$ is the Axiom of Constructability, that says that all the sets of the universe can be build from simpler sets, and it is incompatible with the existence of most large cardinals (LCs).

[2] According to this principle, when comparing two theories the one that can prove more isomorphisms types is preferable, see Maddy, 1997.

[3] For a similar approach, see the *natural conception* of forcing as explained in Hamkins, 2012.

[4] The are also *height extensions* of V, produced by the additions of new sets on top of the hierarchy, but they are not interesting for this particular argument.

the canonical universe.[5]

The problem with this account is that it does not allow to "simulate" two non standard models that are *mutually incompatible*. For example, while it is possible to first force $ZFC+CH$ and then, on top of it, $ZFC+CH+PD$, on the other hand it wouldn't be possible force $ZFC+CH$ and $ZFC+\neg CH$ on top of it (for obvious reasons), or, for a less trivial example, consider $ZFC+V=L$ and $ZFC+\exists$ a measurable cardinal.

For a second example, consider the Axiom of Determinacy. This states that every infinite game is determined, i.e. one of the players has a winning strategy.[6] We know that this axiom is incompatible with the Axiom of Choice. However, if we restrict ourselves to Projective Determinacy that the winning sets, i.e. the victory conditions, are projective sets, then we can force, inside V, $ZFC+PD$. Now, since these infinite games are representable as trees, it would be useful to investigate them with the tools of non-foundational set theory.[7] In particular, we can approach questions from the prospective of extended graphs and their decorations[8]. With this approach, we would define an infinite game as a directed graph (i.e. a tree), and its ground (i.e. the set of its leaves, the bottom-most nodes) as the set of nodes with empty decoration. The following is then true: every extended graph as a unique decoration. In terms of determinacy and games, this means that every game has a unique ground and thus a unique winning winning strategy. However, to get these results, we need to assume the Anti-Foundation Axiom. Consequently, if we believe only in the "simulation theory" sketched above we wouldn't be able to prove these results, since in that case we would have or $ZFC+PD$ or ZFA, but not both at the same time. On the other hand, even in a very simplified toy multiverse composed of only two universes, one well founded and one non-well founded, those results would then be possible.

In summery, the paper argues that the usual argument in favour of the Single Universe, that we can simulate any other universe in it, suffers from very serious limitations. In particular, these simulations cannot be processed at the same time, which in turn makes it impossible to prove a number of important results in set theory - results, however, that by contrast are attainable in a multiverse conception of set theory.

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[5] For a detailed account of forcing, see Nik, 2014.

[6] For details on *AD* see Woodin, 1999.

[7] For an introduction to non-well founded set theory, see Aczel, 1988.

[8] Briefly, a *decoration* is the value of the children of a node, while an *extended graph* is a graph extended with the value of its decorations.