The V-logic Multiverse

de Ceglie, Ternullo

September 2, 2019

In recent years, the notion of 'set-theoretic multiverse' has emerged and progressively gained prominence in the debate on the foundations of set theory. Several conceptions of the set-theoretic multiverse have been presented so far, all of which have advantages and disadvantages. Hamkins' *broad multiverse* ([4]), consisting of *all* models of *all* collections of set-theoretic axioms, is philosophically robust, but mathematically unattractive, as it may fail to fulfil fundamental foundational requirements of set theory. Steel's *set-generic multiverse* ([5]) consisting of all Boolean-valued models $V^{\mathbb{B}}$ of the axioms ZFC+Large Cardinals, is mathematically very attractive and fertile, but too restrictive. In particular, it cannot capture *all* possible outer models, focusing only on the set-generic extensions. Finally, Sy Friedman's *hyperuniverse conception* ([2]), although mathematically versatile and foundationally attractive, has the main disadvantage of postulating that V is countable.

In this paper, we introduce a new conception of the set-theoretic multiverse, that is, the 'V-logic multiverse', which expands on mathematical work conducted within the Hyperunuverse Programme ([1], [3]), but also draws on features of the set-generic multiverse, in particular, on Steel's proposed *axiomatisation* of it.

V-logic is an *infinitary* logic (a logic admitting formulas and proofs of infinite length) whose language $\mathcal{L}_{\kappa^+,\omega}$, in addition to symbols already used in first-order logic, consists of κ-many constants $\overline{a}$, one for each set $a \in V$, and of a special constant symbol $\overline{V}$, which denotes V. In V-logic, one can ensure that the statement asserting the consistency of ZFC+ψ, for some set-theoretic statement ψ, is satisfied by some model M, *if and only if* M is an outer model of V. By outer model we mean here: models obtained through *set-forcing*, *class-forcing*, *hyperclass-forcing* and, in general, any model-theoretic technique able to produce *width extensions* of V. Thus, through the choice of suitable consistency statements, we can generate outer models M, endowed with specific features. The V-logic multiverse is precisely the collection of all such outer models of V.

The following observations help illustrate the adequacy of our method to produce a multiverse concept which, in our view, has better prospects than the ones mentioned above:

1. Contrary to the set-generic multiverse, the V-logic multiverse is broad enough to include all kinds of outer models.

2. Contrary to the hyperuniverse conception, the V-logic multiverse does not reduce to a collection of countable transitive models, as V does not need to be taken to be countable.

As it stands, the V-logic multiverse may be used to pursue two fundamental research directions, both of which are ideally aimed at developing an *axiomatic theory* of the multiverse.

One consists in defining the V-logic multiverse of different extensions of ZFC, by taking into account such axioms as AD, PD, large cardinals, $V = L$ and others, and investigating which relationships obtain among all such V-logic multiverses.

The second direction consists in taking V to be approximated by different structures, such as L, L-like models, V_κ, where κ is some large cardinal and investigate, for instance, whether members of the corresponding V-logic multiverses are compatible with each other, and to what extent. For instance, the L-logic multiverse maximises compatibility, but reduces the extent of structural variability among universes, thus reducing the range of alternative *truth outcomes* in the multiverse.

We argue that the V-logic multiverse is both mathematically more fruitful and philosophically robust than all the other multiverse conceptions, and consequently the best candidate to be the foundation of set theory and mathematics.

References

[1] C. Antos, S.-D. Friedman, R. Honzik, and C. Ternullo. Multiverse Conceptions in Set Theory. *Synthese*, 192(8):2463–2488, 2015.

[2] T. Arrigoni and S. Friedman. The Hyperuniverse Program. *Bulletin of Symbolic Logic*, 19(1):77–96, 2013.

[3] S. Friedman. Evidence for Set-Theoretic Truth and the Hyperuniverse Programme. *IfCoLog Journal of Logics and their Applications*, 3(4):517–555, 2016.

[4] J. D. Hamkins. The Set-Theoretic Multiverse. *Review of Symbolic Logic*, 5(3):416–449, 2012.

[5] J.R. Steel. Gödel's Program. In J. Kennedy, editor, *Interpreting Gödel. Critical Essays*, pages 153–179. Cambridge University Press, Cambridge, 2014.